Infinite permutations vs. infinite words

Anna Frid

Sobolev Institute of Mathematics, Novosibirsk, Russia

12 September 2011

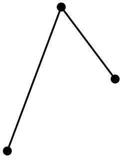
$a = a_0, a_1, a_2, \ldots$ $b = b_0, b_1, b_2, \ldots$ - sequences of reals

$$a \sim b : a_i < a_j \iff b_i < b_j$$

A (finite or infinite) *permutation* is an equivalence class $\alpha = \overline{a} = \overline{b}$.

a and b are *representatives* of α.

Example: $\overline{-100, 200, 197} = \overline{99, 100, 98} = \overline{1, 3, 2}$

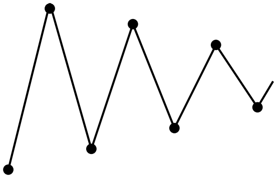


Some authors write just $\alpha = 1\ 3\ 2$. I prefer to distinguish permutations and their representatives.

Infinite permutations

$$a_n = (-1)^{n+1}\frac{1}{2^n}$$

$$\alpha = \overline{a}$$



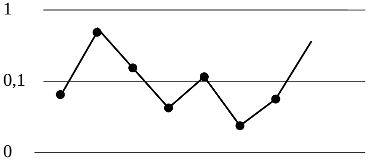
Here there is no representative on integers!

This permutation is *2-periodic* since $\alpha_i < \alpha_j \iff \alpha_{i+2} < \alpha_{j+2}$.

$w_{TM} = 0110100110010110\cdots$ — Thue-Morse word

Consider the sequence a_{TM} of shifts

$.01101001\cdots$,
$.11010010\cdots$,
$.10100110\cdots$,
$.01001100\cdots$,
$.10011001\cdots$,
$.00110010\cdots$, etc.



$$\alpha_{TM} = \overline{a_{TM}}$$

It is natural to consider permutations generated by *binary* words.

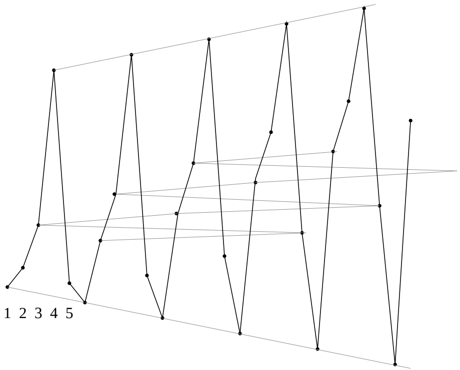
Plan of the talk

- Permutations and their “word” properties:
  - Periodicity
  - Complexities
  - Automatic properties
- Permutations generated by binary words
  - What permutations appear like that?
  - Sturmian permutations
  - Morphic permutations
- Open directions

My coauthors: S. Avgustinovich, D. Fon-Der-Flaass, T. Kamae, P. Salimov, L. Zamboni
Other authors: M. Makarov, A. Valyuzhenich, S. Widmer

Periodic permutation

A t-periodic permutation: $\alpha_i < \alpha_j \Longleftrightarrow \alpha_{i+t} < \alpha_{j+t}$
For $t > 1$, there is a countable number of distinct t-periodic permutations.



The code of this permutation:
$[1, >][2, 5(2), 3(-1), <][4, <]$.

[Fon-Der-Flaass, F., 2005]

Fine and Wilf theorem

Theorem (Fine and Wilf)

If a word of length at least $p+q-(p,q)$ is p-periodic and q-periodic, then it is (p,q)-periodic.

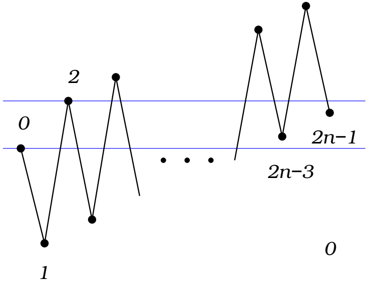
Theorem

If a permutation α of length at least $p+q$ is p-periodic and q-periodic, where $(p,q)=1$, then α is 1-periodic, that is, monotonic.

Monotonic permutation: $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$ or $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$.

BUT!

Fine and Wilf - 2

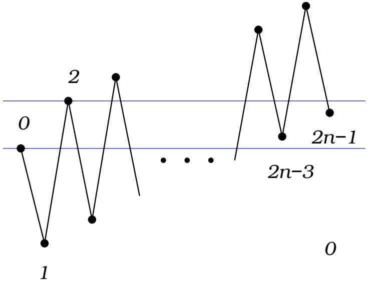


Arbitrarily long permutation which is 4- and 6-periodic but not 2-periodic

Theorem

Suppose that a finite permutation α of length n is p-periodic and q-periodic. Then each its factor of length at most $n - p - q + 2(p, q) + 1$ is (p, q)-periodic.

Local and global periodicity

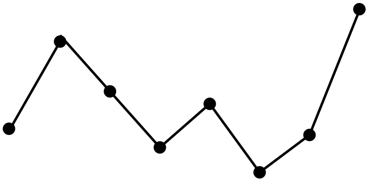


For permutations, local periodicity does not imply global periodicity.
Nothing similar to the critical factorization theorem is possible.

Factors

What is a factor?

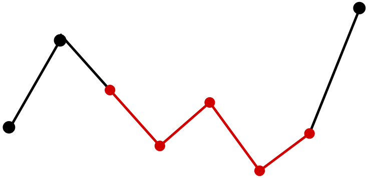
$$011010011001\cdots$$



The number of distinct factors of length n is called the *complexity* of a word or a permutation.

What is a factor?

$01\textcolor{red}{10100}11001\cdots$



The number of distinct factors of length n is called the *complexity* of a word or a permutation.

Low complexity

Theorem

An infinite word/permutation is ultimately periodic if and only if its complexity is bounded.

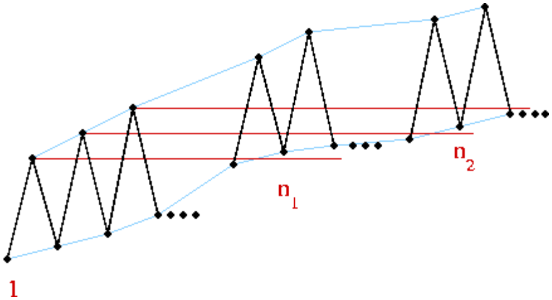
Theorem

The complexity of a non-periodic word is at least $p_w(n) = n + 1$.

Low complexity-2

Theorem (Fon-Der-Flaass, F., 2005)

The complexity of a non-periodic permutation can be arbitrarily low.



Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$01001010100101 \cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max\limits_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$0100101010100101 \cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max\limits_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$0\color{red}{1}0\color{red}{0}10\color{red}{1}0100101\cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$0100101010100101\cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p^*_u(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$010010101001 01 \cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$0100101010 0101 \cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$01001010100101\cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p^*_u(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$$0100101\textcolor{red}{1}0\textcolor{red}{1}00\textcolor{red}{1}01 \cdots \qquad T = (0, 2, 5)$$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p_u^*(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$0100101\textcolor{red}{0}1\textcolor{red}{0}01\textcolor{red}{0}1\cdots \qquad T = (0, 2, 5)$

Maximal pattern complexity

$T = \{0, m_1, \ldots, m_{k-1}\}$ - a *k-window*.

$u = u_0 u_1 \cdots u_n \cdots$ - a word or a permutation;

$u_{n+T} = u_n u_{n+m_1} \cdots u_{n+m_{k-1}}$ - its *T-factor*;

$p_u(T) = \{u_{n+T} | n = 0, 1, \ldots\}$ - the T-complexity of u;

$\max_{|T|=k} p_u(T) = p^*_u(n)$ - *maximal pattern complexity* of u.

For words [Kamae, Zamboni, 2002];
for permutations [Avgustinovich, F., Kamae, Salimov, 2011].

$0100101010010 1\cdots \qquad T = (0, 2, 5)$

Low m. p. complexity

Theorem (Kamae, Zamboni, 2002)

An infinite word w is not ultimately periodic if and only if $p_w^*(n) \geq 2n$ for some n.

Words of complexity $2n$: all Sturmian words, some Toeplitz words, and others.

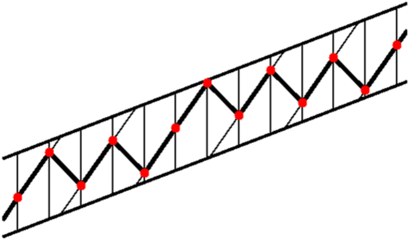
Theorem (Avgustinovich, F., Kamae, Salimov, 2011)

An infinite permutation w is not ultimately periodic if and only if $p_w^*(n) \geq n$ for some n.

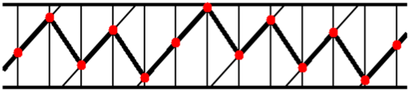
Permutations of complexity n: *exactly* Sturmian permutations.

Sturmian permutations

Representatives generated by Sturmian words:



0 1 0 1 0 0 1 0 1 0 1 0



We fix $x, y > 0$ such that

$$a_{i+1} = \begin{cases} a_i + x, & \text{if } w_i = 0, \\ a_i - y, & \text{if } w_i = 1. \end{cases}$$

Particular case:
$x = \sigma$, $y = 1 - \sigma$

[Makarov, 2006]: It is exactly the permutation generated by this Sturmian word.

Sturmian permutations vs. Sturmian words

	Sturmian words	Sturmian permutations
factor complexity	$n+1$ [classical]	n [Makarov, 2006]
max. p. complexity	$2n$ [Kamae. Zamboni, 2002]	n [Makarov, 2006]
arithmetical complexity: $\#\{u_k u_{k+d} \cdots u_{k+(n-1)d}\}$	$\leq (n-1)n(n+1)/6 + \sum_{p=1}^{n-1}(n-p)\varphi(p) + 2$ [Cassaigne, F., 2007]	$n \sum_{r=1}^{n-1} \varphi(r)$ [Makarov, 2006]
	$(1/6 + 1/\pi^2)n^3 + O(n^2)$	$(3/\pi^2)n^3 + O(n^2)$

Quadratic permutations vs. quadratic words

Let us fix some irrational α and consider the sequence a of fractional parts

$$a_n = \{n^2\alpha\}.$$

Consider

- The permutation $\alpha = \overline{a}$: its factor complexity is $O(n^4)$ [F., 2012]
- The sequence b on $\{0, 1\}$ defined by

  $$b_i = \lfloor 2a_i \rfloor.$$

  Its factor complexity is $O(n^3)$ [Belov, Kondakov, 1995; see also Arnoux, Mauduit, 1996]

Thue-Morse permutation

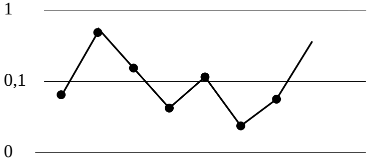
Thue-Morse word: fixed point

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001 \cdots$$

of the morphism

$$\varphi : \begin{cases} 0 \to 01, \\ 1 \to 10. \end{cases}$$

Corresponding permutation:

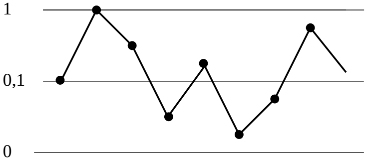


Directly as the fixed point of the morphism $\varphi : [-1,1] \to [-1,1]$:

$$\varphi(x) = \begin{cases} \frac{1}{2}x, \frac{1}{2}x - 1, & \text{if } x > 0; \\ \frac{1}{2}x, \frac{1}{2}x + 1, & \text{if } x \leq 0 \end{cases}$$

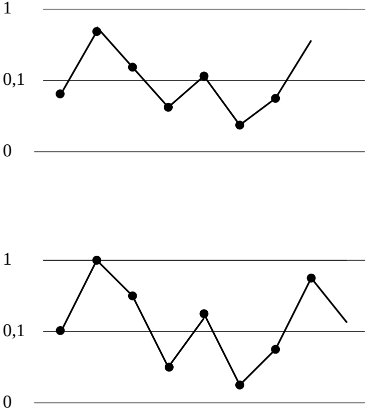
$$0,\ 1,\ \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{1}{8}, -\frac{7}{8}, -\frac{3}{8}, \frac{5}{8}, -\frac{1}{8}, \frac{7}{8}, \frac{3}{8}, -\frac{5}{8}, \ldots$$

[Makarov, 2009]



Thue-Morse permutation-3

Two representatives are different!



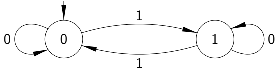
Two remarks

- Factor complexities of the Thue-Morse word [Brlek; de Luca, Varricchio, 1989] and of the Thue-Morse permutation [Widmer, 2011] are both linear but different.
- It would be nice to find more beautiful morphisms defining permutations.

Automatic words

$$w_{TM} = 0110\ 1001\ 1001\ 0110 \cdots$$

nth symbol of the Thue-Morse word = number of 1s (modulo 2) in the binary representation of n.



A k-automatic word:

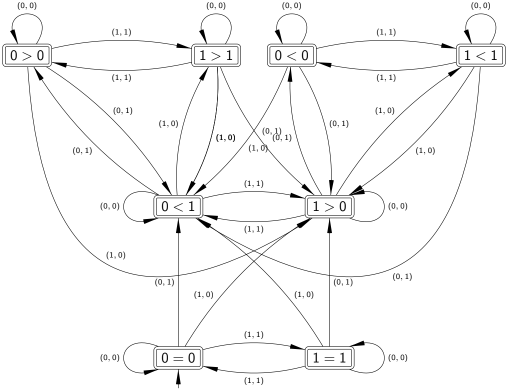
- nth symbol are obtained by a finite automaton from the k-ary representation of n;
- image of a fixed point of a k-uniform morphism on a (finite but possibly big) alphabet under a coding;
- several other characterizations.

Automatic permutations?

We feed to an automaton the k-ary symbols of a pair (i, j) and get as the output the information if $\alpha_i < \alpha_j$ or $\alpha_i > \alpha_j$ or $\alpha_i = \alpha_j$ ($\Longleftrightarrow i = j$).

Example

To compair elements no. 3 and 5 of a 2-automatic permutation we feed to the automaton the pairs $(0, 1), (1, 0), (1, 1)$ and get as output one of the symbols $<$ or $>$.



0 > 0
1 > 1
0 < 0
1 < 1
0 < 1
1 > 0
0 = 0
1 = 1
(0, 0)
(1, 1)
(0, 1)
(1, 0)

Automatic words give automatic permutations

Theorem (F.,Zamboni,2011)

A permutation generated by a k-automatic word is k-automatic.

But in general, the number of states of its automaton is bounded just by $O(d^4!)$, where d is the cardinality of the (possibly BIG) alphabet of the fixed point.
For Thue-Morse, the general construction would give 16! states, not just 8.

What else have I forgotten?

[Makarov, 2006]: the number of all permutations of length n that appear from binary words is

$$P(n) = \sum_{t=1}^{n-1} \psi(t) \cdot 2^{n-1-t},$$

where

$$\psi(t) = \sum_{d|t} \mu(t/d) \cdot 2^d$$

is the number of primitive words of length t.

$$P(n+1) = 2^n(n - \alpha + O(n2^{-n/2}); \ \alpha = 1.3827 \cdots$$

The same function had appeared in [Domaratzki, Kisman, Shallit, 2002] as the number of languages accepted by finite automata with n states.

Possible techniques

Makarov worked with different types of left "special" permutations: he distinguishes *binary* and *strange* permutation factors.

He found [2009] the complexity of the *period doubling* permutation generated by the word

$$0100010101000100010001\cdots,$$

but it was S. Widmer [2011] who proved the formula for the Thue-Morse permutation!

A. Valyuzhenich has generalized his result by just accurate counting of permutations which are descendants of a given short one.

Questions

- So, what is the best technique(s) to work with permutations generated by words?
- What about words over greater alphabets?
- Is there a theory arising from morphisms on permutations?
- What are "natural" properties for permutations?
- What other new interesting permutations are worth considering?

THANK YOU