Permutation Complexity and the Letter Doubling Map

WORDS 2011

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September 12, 2011

What is a word?

Definition

A **word** is a finite, (right) infinite, or biinfinite sequence of symbols taken from a finite non-empty set, $\mathcal{A}$, called the **alphabet**.

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Given a word $\omega = a_1 a_2 a_3 \ldots$, any finite subword of the form $u = a_i a_{i+1} \ldots a_{i+n-1}$ is called a **factor** of ω of **length** $n \geq 1$, denoted $|u| = n$.

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Periodic Words

- An infinite word ω is said to be **eventually periodic** if $\omega = vuuuu\ldots$, for some finite v and u, with u non-empty.
- A word that is not eventually periodic is called **aperiodic**.

Binary Aperiodic Words

- 1.0111010100000100111 ...
- 11.00100100001111110110 ...

Binary Aperiodic Words

- $1.0110101000001001111\ldots = \sqrt{2}$

- $11.00100100001111110110\ldots = \pi$

Fibonacci

Fibonacci word:

$$\mathbf{f} = \varphi(\mathbf{f}) = 0100101001001010010 1\ldots$$

Fixed point of the morphism $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$

$$0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto 01001010 \mapsto \cdots$$

Thue-Morse

Thue-Morse word:

$$\mathbf{T} = \phi(\mathbf{T}) = 0110100110010110 1001\ldots$$

Fixed point of the morphism $\phi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}$

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto \cdots$$

Factor Complexity

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Factors

Given an infinite word $\omega = a_1 a_2 a_3 \ldots$, let $\mathcal{F}_\omega(n)$ denote the set of factors of ω of length n:

$$\mathcal{F}_\omega(n) = \{ a_i a_{i+1} \ldots a_{i+n-1} \,|\, i \geq 1 \} .$$

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Factor Complexity

The function $\rho_\omega(n) = Card\,(\mathcal{F}_\omega(n))$ is called the **factor complexity function**.

Periodicity and Factor Complexity

Theorem. (M. Morse and G.A. Hedlund, 1938)

An infinite word ω is eventually periodic if and only if $\rho_\omega(n) \leq n$ for some $n \geq 1$. A biinfinite word ω is periodic if and only if $\rho_\omega(n) \leq n$ for some $n \geq 1$.

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Aperiodic $\iff$ $\rho_\omega(n) \geq n + 1$ for each n.

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Sturmian Words

Let ω be an infinite word. Then ω is **Sturmian** if and only if $\rho_\omega(n) = n+1$ for each n.

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More formally, an infinite permutation is the ordered triple $\pi = \langle \mathbb{N}, \prec_\pi, < \rangle$, where $\prec_\pi$ and $<$ are linear orders on $\mathbb{N}$.

Permutation Complexity

Subpermutation

For $a < b$, the **subpermutation** $\pi[a, b]$ of π is the permutation of $\{1, 2, \ldots, b - a + 1\}$ so that for each $0 \leq i, j \leq (b - a)$,

$$\pi[a, b](i) < \pi[a, b](j) \iff \pi(a + i) < \pi(a + j)$$

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Set of Subpermutations

$$\mathrm{Perm}_{\pi}(n) = \{ \pi[a, a + n - 1] \in S_n \mid a \in \mathbb{N} \}$$

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Permutation Complexity

The **permutation complexity function** of an infinite permutation is

$$\tau_\pi(n) = Card(\text{Perm}_\pi(n))$$

Words and Permutations

Infinite Permutation of a Word

Let $\omega = \omega_0\omega_1\omega_2\ldots$ be an infinite aperiodic word, and $<$ a linear ordering of $\mathcal{A}$. Then ω defines an infinite permutation π_ω defined by

$$\pi_\omega(i) < \pi_\omega(j) \iff \omega_i\omega_{i+1}\omega_{i+2}\ldots < \omega_j\omega_{j+1}\omega_{j+2}\ldots$$

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The **permutation complexity of** ω is

$$\tau_\omega(n) = Card(\mathrm{Perm}_{\pi_\omega}(n))$$

Permutation and Subpermutation

Thue-Morse word:

$$\mathbf{T} = 0110100110010110100 1\ldots$$

$$1100101\ldots < 1101001\ldots \quad \text{so} \quad \pi_T(7) < \pi_T(1)$$

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$$1100101\ldots < 1101001\ldots \quad \text{so} \quad \pi_T(7) < \pi_T(1)$$

$$\pi_T[2,4] = (3\;1\;2)$$

$$0100110\ldots < 1001100\ldots < 1010011\ldots$$

Permutation Complexity

Thue-Morse word:

$$\mathbf{T} = 0110100110010110 1001\ldots$$

$\mathrm{Perm}_T(4)$

(1 2 4 3) (1 3 4 2) (2 4 1 3) (2 4 3 1)
(3 1 2 4) (3 1 4 2) (4 2 3 1) (4 3 1 2)

$$\tau_T(4) = 8$$

The Doubling Map

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Letter Doubling Map

For each letter a in the alphabet $\mathcal{A}$,

$$d(a) = aa$$

Uniform Recurrence

Recurrent Words

A word ω is **recurrent** if each factor of ω occurs infinitely often in ω.

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Uniformly Recurrent Binary Words

For a uniformly recurrent binary word ω over $\mathcal{A}$, there is a k so that for $a, b \in \mathcal{A}$, $a \neq b$, a^k is a factor of ω, but a^{k+1} and b^{k+1} are not.

Image of Binary Uniformly Recurrent Words

Thus if ω is uniformly recurrent, there is a positive integer M so that for each factor V of ω with $|V| \geq M$, $\mathcal{F}_\omega(k) = \mathcal{F}_V(k)$.

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Lemma

Let ω be a binary uniformly recurrent word. For $n \geq M$:

$$\tau_{d(\omega)}(2n-1) \leq 2(\tau_\omega(n+k))$$
$$\tau_{d(\omega)}(2n) \leq \tau_\omega(n+k) + \tau_\omega(n+k+1)$$

Image of Sturmian

Theorem. (E. Coven, G. Hedlund, 1973)

Sturmian words are uniformly recurrent.

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Sturmian words are uniformly recurrent.

Theorem

Let s be a Sturmian word over $\mathcal{A}$, where for $a, b \in \mathcal{A}$, $a \neq b$, there are strings of either k or $k-1$ a's between each b, with $k > 1$. There is an M so that each factor of s of length at least M will contain each of a^k, $a^{k-1}b$, $\ldots$, ab, b. For each $n \geq 2M$ the permutation complexity of $d(s)$ is

$$\tau_{d(s)}(n) = n + 2k + 1$$

Image of Fibonacci

$$\mathbf{f} = 01001010010010100101\ldots$$

$$\mathbf{d(f)} = 0011000011001100001100\ldots$$

$$k = 2$$

Image of Fibonacci

$$\mathbf{f} = 0100101001001010010 1\dots$$

$$\mathbf{d(f)} = 0011000011001100001100\dots$$

$$k = 2$$

Permutation Complexity of d(f)

For $n \geq 9$:

$$\tau_{d(f)}(n) = n + 2 \cdot (2) + 1 = n + 5$$

Permutation Complexity of the Thue-Morse Word

Thue-Morse Word

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Theorem

For any $n \geq 5$, where $n = 2^r + p$ with $0 < p \leq 2^r$,

$$\tau_T(n) = 2(2^{r+1} + p - 2)$$

Image of Thue-Morse

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Theorem

For the Thue-Morse word T, let $n \geq 9$.

- If $n = 2^r$, then
$$\tau_{d(T)}(2n-1) = 2^{r+2} + 2^{r+1}$$
$$\tau_{d(T)}(2n) = 2^{r+2} + 2^{r+1} + 4$$
- If $n = 2^r + p$ for some $0 < p \leq 2^r - 1$, then
$$\tau_{d(T)}(2n-1) = 2^{r+3} + 4p$$
$$\tau_{d(T)}(2n) = 2^{r+3} + 4p + 2$$

Thank you.