Substitutions over infinite alphabet generating $(-\beta)$-integers

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b-expansions

Let $b \in \mathbb{R}$, $|b| > 1$, $x \in [l, l+1)$, $l \in \mathbb{R}$. The *b*-expansion of x is the sequence $(x_i)_{i \geq 1}$,

$$x = \frac{x_1}{b} + \frac{x_2}{b^2} + \frac{x_3}{b^3} + \dots ,$$

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where

$$x_i := \lfloor bT^{i-1}(x) - l \rfloor ,$$
$$T(x) := bx - \lfloor bx - l \rfloor .$$

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where

$$\begin{aligned} x_i &:= \lfloor bT^{i-1}(x) - l \rfloor , \\ T(x) &:= bx - \lfloor bx - l \rfloor . \end{aligned}$$

We write $d(x) := x_1 x_2 x_3 \cdots$.

b-expansions $\cdots$ choice of b, l

- Rényi:
  $b = \beta > 1$, $[l, l+1) = [0, 1)$,
  $x_i \in \{0, 1, \cdots, \lceil\beta\rceil - 1\}$

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- Akiyama, Scheicher:
  $b = \beta > 1$, $[l, l+1) = \left[-\frac{1}{2}, \frac{1}{2}\right)$,
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- Ito, Sadahiro:
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- ADMP:
  $b = -\beta < -1$, $l \in (-1, 0]$,
  $x_i \in \{\lfloor -l(\beta+1) - \beta \rfloor, \ldots, \lfloor -l(\beta+1) \rfloor\}$

b-expansions $\cdots$ admissibility

$(x_i)_{i\in\mathbb{N}}$ is (b,l)-admissible if $d(x) = x_1x_2\cdots$ for some $x \in [l, l+1)$

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Theorem

For fixed b, l it holds: $(x_i)_{i\in\mathbb{N}}$ is (b,l)-admissible iff

$$d(l) \preceq_{(alt)} x_ix_{i+1}x_{i+2}\cdots \prec_{(alt)} d^*(l+1) = lim_{\epsilon\to 0+} d(l+1-\epsilon)\,,$$

where

- $b = \beta > 1 \to$ *lexicographic ordering* $\preceq$
- $b = -\beta < -1 \to$ *alternate ordering* $\preceq_{alt}$

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Note:

$$u_1u_2\cdots \prec_{alt} v_1v_2\cdots \text{ if } (-1)^k(u_k-v_k) < 0,\ k = \min\{i \geq 1, u_i \neq v_i\}$$

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$(-\beta)$-expansions of reals

$l \in (-1, 0] \Rightarrow$ existence of $\langle x \rangle_{-\beta}$ for all $x \in \mathbb{R}$:

$\langle x \rangle_{-\beta} := x_1 x_2 \cdots x_k \bullet x_{k+1} x_{k+2} \cdots$, where

$$\frac{x}{(-\beta)^k} \in [l, l+1) \quad \text{and} \quad d\Big(\frac{x}{(-\beta)^k}\Big) = x_1 x_2 x_3 \cdots$$

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Example

Let $\beta = 1+\sqrt{2}$ *root of* x^2-2x-1, $[l, l+1) = \left[-\frac{\beta^9}{\beta^9+1}, \frac{1}{\beta^9+1}\right)$.

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Let $\beta = 1 + \sqrt{2}$ *root of* $x^2 - 2x - 1$, $[l, l+1) = \big[-\frac{\beta^9}{\beta^9+1}, \frac{1}{\beta^9+1}\big)$.
$\langle 1 \rangle_{-\beta}$ *can be:*

$$1\bullet 0^\omega = 120\bullet 0^\omega = 13210\bullet 0^\omega = 1322210\bullet 0^\omega = 132222210\bullet 0^\omega$$

$(-\beta)$-expansions of reals $\cdots$ uniqueness

- $I \in \big(-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\big]$ $\cdots$ unique $\langle x \rangle_{-\beta}$ for all $x \in \mathbb{R}$

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Solution:

- unique $\langle x\rangle_{-\beta} = x_k \cdots x_0 \bullet x_{-1} \cdots$, such that $0x_kx_{k-1}\cdots$ is admissible

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- $l \in \left(-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$ $\cdots$ unique $\langle x \rangle_{-\beta}$ for all $x \in \mathbb{R}$
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- $\langle x \rangle_{-\beta}$,$\langle (-\beta)^k x \rangle_{-\beta}$ use the same digit string

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$(-\beta)$-integers $\cdots$ definition

We define

$$\mathbb{Z}_{-\beta} := \left\{ x = \sum_{i=0}^{k} x_i(-\beta)^i \;\middle|\; 0x_k x_{k-1} \cdots x_1 x_0 0^\omega \text{ is admissible} \right\}.$$

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- definition valid for $l \in (-1, 0]$
- $\mathbb{Z}_{-\beta} \subseteq \bigcup_{i \geq 0} (-\beta)^i T^{-i}(0)$
- if $l \in \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1} \right]$, the equality holds

$(-\beta)$-integers $\cdots$ distances I

$$\mathcal{S}(k) := \{a_{k-1}a_{k-2}\cdots a_0 0^\omega \mid a_{k-1}a_{k-2}\cdots a_0 0^\omega \text{ is admissible}\},$$

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- $\gamma(a_{k-1}\cdots a_1 a_0) := a_{k-1}(-\beta)^{k-1} + \cdots + a_1(-\beta) + a_0$

$(-\beta)$-integers $\cdots$ distances II

Proposition

Let x,y be neighbours in $\mathbb{Z}_{-\beta}$ and $\langle x\rangle_{-\beta} = x_n x_{n-1}\cdots x_0 \bullet 0^\omega$, $\langle y\rangle_{-\beta} = y_m y_{m-1}\cdots y_0 \bullet 0^\omega$ with n, m maximal possible.

Their distance is equal to

$$\Delta_k := \Big|(-\beta)^k + \gamma\big(min(k)\big) - \gamma\big(max(k)\big)\Big|,$$

where k is maximal such that $x_k \neq y_k$.

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- necessary to know $min(k)$, $max(k)$
- hard to get an explicit general formula

$(-\beta)$-integers $\cdots$ extremal strings

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Denote $d(l) = l_1 l_2 l_3 \cdots$, $d^*(l+1) = r_1 r_2 r_3 \cdots$.

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- *for* $k \geq 1$ *either* $min(k) = l_1 l_2 \cdots l_k$ *or exists* $m(k) \in \{0, \cdots, k-1\}$ *such that*

$$min(k) = \begin{cases} l_1 l_2 \cdots (l_{k-m(k)}+1) min(m(k)) & \text{if } k-m(k) \text{ even} \\ l_1 l_2 \cdots (l_{k-m(k)}-1) max(m(k)) & \text{if } k-m(k) \text{ odd} \end{cases}$$

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- *for* $k \geq 1$ *either* $max(k) = r_1 r_2 \cdots r_k$ *or exists* $m'(k) \in \{0, \cdots, k-1\}$ *such that*

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coding $\mathbb{Z}_{-\beta}$ by infinite word $\cdots$ introduction

coding distances in $\mathbb{Z}_{-\beta}$ by integers $\rightarrow$ bidirectional infinite word

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coding distances in $\mathbb{Z}_{-\beta}$ by integers $\to$ bidirectional infinite word

- Steiner: properties of $T \to$ antimorphism over some alphabet for case $l = -\frac{\beta}{\beta+1}$ (finite alphabet if $d(l)$ ev. periodic)

coding $\mathbb{Z}_{-\beta}$ by infinite word $\cdots$ introduction

coding distances in $\mathbb{Z}_{-\beta}$ by integers $\rightarrow$ bidirectional infinite word

- Steiner: properties of T $\rightarrow$ antimorphism over some alphabet for case $l = -\frac{\beta}{\beta+1}$ (finite alphabet if $d(l)$ ev. periodic)
- ADMP: combinatorial approach $\rightarrow$ antimorphism over $\mathbb{N}$ for $l \in (-1, 0]$,
  several choices of l, ev. periodic $d(l)$ $\rightarrow$ projection into finite alphabet

coding $\mathbb{Z}_{-\beta}$ by infinite word I

Let $(z_n)_{n\in\mathbb{Z}}$ be strictly increasing and

$$z_0 = 0 \quad \text{and} \quad \mathbb{Z}_{-\beta} = \{z_n \mid n \in \mathbb{Z}\}.$$

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- for any $n \in \mathbb{Z}$ exists unique $k \in \mathbb{N}$ such that

$$z_{n+1} - z_n = \left|(-\beta)^k + \gamma\big(min(k)\big) - \gamma\big(max(k)\big)\right|,$$

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- put $v_n := k$

coding $\mathbb{Z}_{-\beta}$ by infinite word II

Theorem

Let $\mathbf{v}_{-\beta}$ be the word associated with $\mathbb{Z}_{-\beta}$. There exists an antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\Psi = \Phi^2$ is a non-erasing non-identical morphism and $\Psi(\mathbf{v}_{-\beta}) = \mathbf{v}_{-\beta}$

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Φ is always of the form

$$\Phi(2l) = S_{2l}(2l+1)\widetilde{R_{2l}} \quad \text{and} \quad \Phi(2l+1) = R_{2l+1}(2l+2)\widetilde{S_{2l+1}}\,,$$

where words R_j, S_j depend only on j and on $min(k), max(k)$ with $k \in \{j, j+1\}$.

coding $\mathbb{Z}_{-\beta}$ by infinite word $\cdots$ example infinite

β root of $x^2 - 4x + 2$:

	$\beta,\ l = 0$	$-\beta,\ l = -\frac{\beta}{\beta+1}$	$-\beta,\ l = -\frac{1}{2}$
ref. strings	$\cdots 31^\omega$	$(32)^\omega \cdots 0(32)^\omega$	$210^\omega \cdots \overline{21}0^\omega$

coding $\mathbb{Z}_{-\beta}$ by infinite word $\cdots$ example infinite

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	$\Delta_1 = \beta - 3$	$\Delta_1 = \beta - 2$	$\Delta_1 = \beta - 2$
(anti)morphism	$0 \to 0^3 1$	$0 \to 0^2 1$	$0 \to 010$
	$1 \to 01$	$1 \to 0^2 21$	$1 \to 0210$
		$2 \to 0^2 31$	$2 \to 3210$
		$3 \to 0^2 41$	$3 \to 014$
		$4 \to 0^2 51$	$4 \to 510$
		$5 \to 0^2 61$	$5 \to 016$
		$\cdots$	$\cdots$

coding $\mathbb{Z}_{-\beta}$ by infinite word $\cdots$ example finite

β root of $x^2 - 4x + 2$:

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	$1 \to 01$	$1 \to 0^2 1^2$	$1 \to 01^2 0$