Systems of Word Equations and Polynomials: a New Approach

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Word Equations

- If u, v are words over the unknowns $x, y, z, \ldots$, then $u = v$ is a *word equation*.
- If a substitution $x = \alpha, y = \beta, z = \gamma, \ldots$ makes u and v equal, then it is a *solution*.
- A solution is *periodic*, if the values of the unknowns are powers of a common word.

Example

Consider the word equation $xyz = zyx$.

- $x = aba, y = b, z = a$ is a (nonperiodic) solution, because $aba \cdot b \cdot a = a \cdot b \cdot aba$.
- $x = ab, y = (ab)^2, z = \varepsilon$ is a periodic solution.

Independent Systems

A system of equations is *independent*, if it is not equivalent with any of its subsystems.

Open problems:

- Are there independent systems of three equations on three unknowns having a nonperiodic solution?
- Are there unboundedly large such systems?

We prove that there is a bound depending on the length of the shortest equation.

Length Types

The *length type* of a solution (x, y, z) is the vector $(|x|, |y|, |z|)$.

Example

The nonperiodic solutions of $xyz = xzy, xyxzyz = zxzyxy$ are

$$x = p, y = q, z = \varepsilon \qquad \text{or} \qquad x = p, y = q, z = pq.$$

The length types $(|x|, |y|, |z|)$ satisfy

$$|z| = 0 \qquad \text{or} \qquad |x| + |y| = |z|$$

We prove that the length types of nonperiodic solutions of a pair of nonequivalent equations on three unknowns are covered by finitely many two-dimensional subspaces.

Polynomials

Let the alphabet be $\Sigma = \{1, 2, \ldots, m\}$. For a word $w = a_0 \ldots a_{n-1} \in \Sigma^n$ we define a polynomial

$$P_w = a_0 + a_1 X^1 + \cdots + a_{n-1} X^{n-1}.$$

Example

If $w = 1212$, then $P_w = 1 + 2X + X^2 + 2X^3$.

Solutions of Fixed Length: Example

Example

The solutions of $xyz = zxy$ of length type $L = (1, 1, 2)$ are determined by

$$P_{xyz} = P_{zxy}$$

or

$$P_x + XP_y + X^2 P_z = P_z + X^2 P_x + X^3 P_y$$

or

$$(1 - X^2) \cdot P_x + (X - X^3) \cdot P_y + (X^2 - 1)P_z = 0.$$

One solution is $P_x = 1, P_y = 2, P_z = 1 + 2X$, that is $x = 1, y = 2, z = 12$.

Solutions of Fixed Length

For a word equation $E : y_1 \dots y_k = z_1 \dots z_l$, a variable x and a length type L, let

$$Q_{E,x,L} = \sum_{y_i = x} X^{|y_1 \dots y_{i-1}|} - \sum_{z_i = x} X^{|z_1 \dots z_{i-1}|}.$$

Theorem

$(x_1, \dots, x_n)$ *of length type* L *is a solution of* E *if and only if*

$$\sum_i Q_{E,x_i,L} P_{x_i} = 0.$$

Q-Theorem

The solutions of length type L of a pair E_1, E_2 are determined by

$$Q_{E_1,x,L} P_x + Q_{E_1,y,L} P_y + Q_{E_1,z,L} P_z = 0,$$
$$Q_{E_2,x,L} P_x + Q_{E_2,y,L} P_y + Q_{E_2,z,L} P_z = 0.$$

- If there is a nonperiodic solution, then the solutions of the linear system of equations form a two-dimensional space.
- Then the linear equations must be dependent.
- Then the word equations have the same solutions of length type L.

Generalization for n unknowns requires the notion of the *rank* of a solution.

Special Case of the Graph Lemma

Consider a pair of equations

$$E_1 : x \cdots = y \cdots , \qquad E_2 : x \cdots = z \cdots$$

on three unknowns and the solutions with length type $L \in \mathbb{N}_1^3$. The rank of the matrix

$$\begin{pmatrix} Q_{E_1,x,L} & Q_{E_1,y,L} & Q_{E_1,z,L} \\ Q_{E_2,x,L} & Q_{E_2,y,L} & Q_{E_2,z,L} \end{pmatrix}$$
$$= \begin{pmatrix} 1 + X(\dots) & -1 + X(\dots) & X(\dots) \\ 1 + X(\dots) & X(\dots) & -1 + X(\dots) \end{pmatrix}$$

is 2, so there are no such nonperiodic solutions.

Theorems

Theorem (Czeizler-Karhumäki-07, can be generalized)

If a pair of nontrivial equations on three unknowns has a nonperiodic solution where no two of the unknowns commute, then there is $k \geq 1$ such that the equations are of the form $x \cdots = y^k z \cdots$.

Theorem (almost the same as Kortelainen-98)

Let $m, n \geq 1$, $s_j, t_j \in \Sigma^*$ and $u_j, v_j \in \Sigma^+$. If

$$s_0 u_1^i s_1 \ldots u_m^i s_m = t_0 v_1^i t_1 \ldots v_n^i t_n.$$

holds for $m + n$ values of i, then it holds for all i.

Finitely Many Subspaces: Example

Example

If the pair $E_1 : xyz = zxy, E_2 : xyxzyz = zxzyxy$ has a nonperiodic solution of length type L, then the vectors $(Q_{E_1,x,L}, Q_{E_1,y,L}, Q_{E_1,z,L})$ and $(Q_{E_2,x,L}, Q_{E_2,y,L}, Q_{E_2,z,L})$ must be dependent. So the determinant

$$\begin{aligned} &Q_{E_1,x,L} Q_{E_2,z,L} - Q_{E_1,z,L} Q_{E_2,x,L} \\ =&X^{2|x|+|y|} + X^{2|x|+2|y|+|z|} + X^{|x|+2|z|} + X^{|x|+|y|+|z|} \\ &- X^{2|x|+|y|+|z|} - X^{|x|+|z|} - X^{2|x|+2|y|} - X^{|x|+|y|+2|z|} \end{aligned}$$

must be zero for $(|x|, |y|, |z|) = L$. Thus L must satisfy one of

$$|z| = 0, \qquad |x| + |y| = |z|, \qquad |y| = 0.$$

Finitely Many Subspaces

Theorem (can be generalized)

Let E_1, E_2 be nonequivalent nontrivial equations on three unknowns.

- *The length types of nonperiodic solutions of the pair are covered by a union of $|E_1|^2$ two-dimensional subspaces of $\mathbb{Q}^3$.*
- *If $V_1, \dots, V_m$ is a minimal such cover and $L \in V_i$ for some i, then E_1 and E_2 have the same nonperiodic solutions of length type L.*

Balanced Equations

An equation $u = v$ is *balanced*, if $|u|_x = |v|_x$ for every unknown x.

Theorem (Harju-Nowotka-03, can be generalized)

Let E_1, E_2 be a pair of equations on three unknowns having a nonperiodic solution. If E_1 is not balanced, then every nonperiodic solution of E_1 is a solution of E_2.

Proof.

The length types of solutions of E_1 are covered by a single two-dimensional space V. Now E_1 and E_2 have the same nonperiodic solutions of length type L for all $L \in V$. □

Independent Systems

Theorem (can be generalized)

If $E_1, \dots, E_m$ is an independent system on three unknowns having a nonperiodic solution, then $m \leq |E_1|^2 + 1$.

Idea of the proof.

- Length types of nonperiodic solutions of E_1, E_2 are covered by $|E_1|^2$ subspaces.
- Length types of nonperiodic solutions of E_1, E_2, E_3 are covered by $|E_1|^2 - 1$ subspaces.
- ...
- Length types of nonperiodic solutions of $E_1, \dots, E_m$ are covered by $|E_1|^2 - m + 2 \geq 1$ subspaces.

□