Optimizing Properties of Balanced Words

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Balanced words and Sturmian sequences

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There are several equivalent definitions of Sturmian sequences. Let $w = w_1 w_2 \dots$ be an infinite 0-1 sequence and put $p_w(n) = \#\{w_j \dots w_{j+n-1} : j \geq 1\}$ – the complexity function of w.

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The 1-ratio $\gamma = \lim_{n\to\infty} |w_1 \dots w_n|_1 / n$ is well defined for any Sturmian sequence w.

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As is well known, there are precisely q balanced words in $\mathcal{W}_{p,q}$, all of which are in the same orbit (= all cyclic permutations of a word), so if $\mathbb{W}_{p,q}$ is defined to be the set of all orbits of words in $\mathcal{W}_{p,q}$, there is a unique balanced orbit in $\mathbb{W}_{p,q}$.

Theorem (O. Jenkinson, 2009)

Suppose $1 \leq p < q$ *are coprime integers. For* $w \in \mathbb{W}_{p,q}$*, the product* $B(w)$ *is maximized precisely when* w *is balanced.*

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Maximizing measures

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$$\mathrm{bar}(\mu) = \int_0^1 x\ d\mu(x),$$

i.e., the barycentre of μ.

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Here S_γ is the following. Let $\varphi : [0, 1) \to [0, 1)$ be defined as follows:

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Then the Sturmian measure S_γ is the push forward of the Lebesgue measure on $[0,1)$ under φ_γ.

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If γ is irrational, then S_γ is supported by a Cantor set. For instance, if $\gamma = \frac{3-\sqrt{5}}{2}$, then $\mathrm{supp}(S_\gamma)$ is the closure of the set of all shifts of the Fibonacci word $f = 0100101001001\ldots$ (It is indeed a Cantor set because f has such a low complexity!)

Growth of matrix products and joint spectral radius

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Definition

A set of matrices $\mathcal{A}$ is said to have the finiteness property if there exists an eventually periodic maximizing sequence for $\mathcal{A}$.

Example

Put $\mathcal{A} = \{A_0, A_1\}$, where

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

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$$\varrho(\mathcal{A}) = (\rho(A_0 A_1))^{1/2} = (1 + \sqrt{5})/2.$$

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Theorem (K. Hare, I. Morris, N. Sidorov and J. Theys, 2011)

There exists a continuum $\mathfrak{C}$ of α with no finiteness property. Moreover, each maximizing sequence for any fixed $\alpha \in \mathfrak{C}$ is "essentially Sturmian".

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Each element of $\mathfrak{C}$ can be written in a completely explicit form (an infinite product).

Non-Sturmian maximizing sequences

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Open question. Is there a pair of matrices such that any maximizing sequence for this pair is of an exponential complexity?