INFORMATION THEORY: SOURCES, DIRICHLET SERIES, REALISTIC ANALYSIS OF DATA STRUCTURES

Mathieu Roux and Brigitte Vallée
GREYC Laboratory
(CNRS and University of Caen, France)

Talk also based on joint works with
Viviane Baladi, Eda Cesaratto, Julien Clément,
Jim Fill, Philippe Flajolet

WORDS 2011, Prague, September 2011

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– unifies the analyses for text algorithms and searching/sorting algorithms

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– defines a natural subclass of sources, the dynamical sources
– provides sufficient conditions for tameness of dynamical sources

– provides probabilistic analyses for data structures built on tame sources.

Plan of the talk.

– General motivations: Dirichlet generating functions and tameness

– An important class of sources: dynamical sources.

– Tameness in the case of dynamical sources

– Conclusion and possible extensions.

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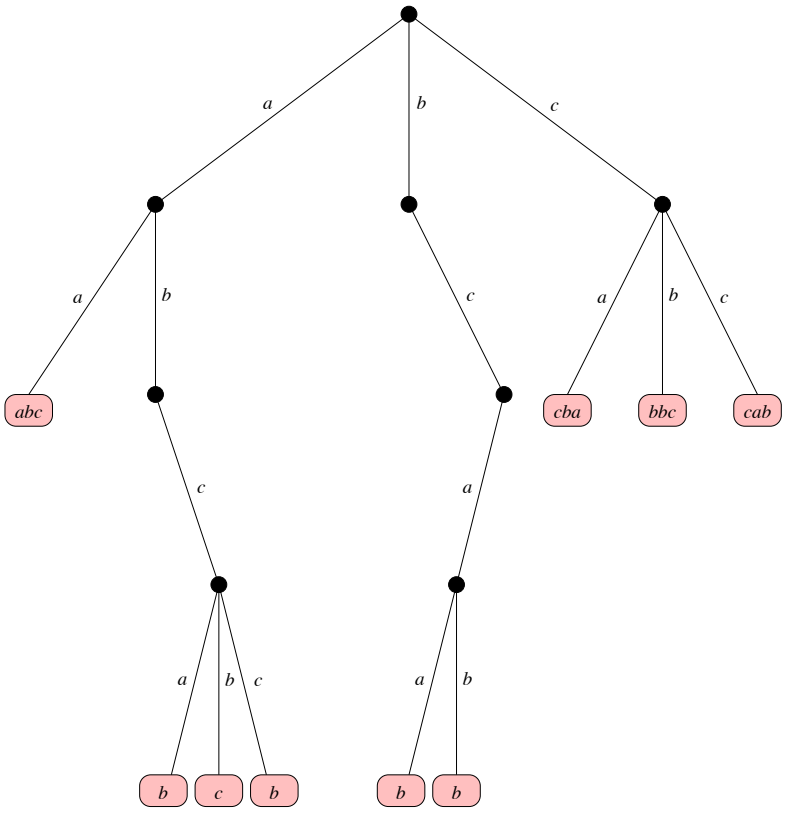
– for comparing two words, importance of the structure of words

– for comparing two keys, transparence of the structure of keys

only their relative order plays a role.

Text algorithms and dictionaries : The trie structure

Probabilistic study



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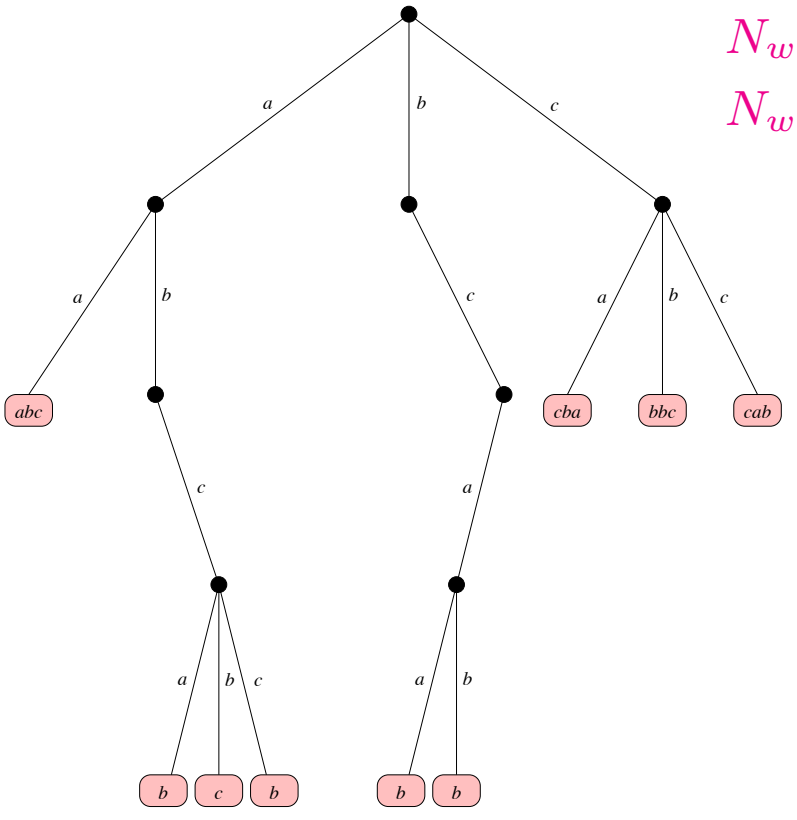

Main parameter on a node n_w labelled with prefix w:

N_w := the number of words which begin with prefix w.

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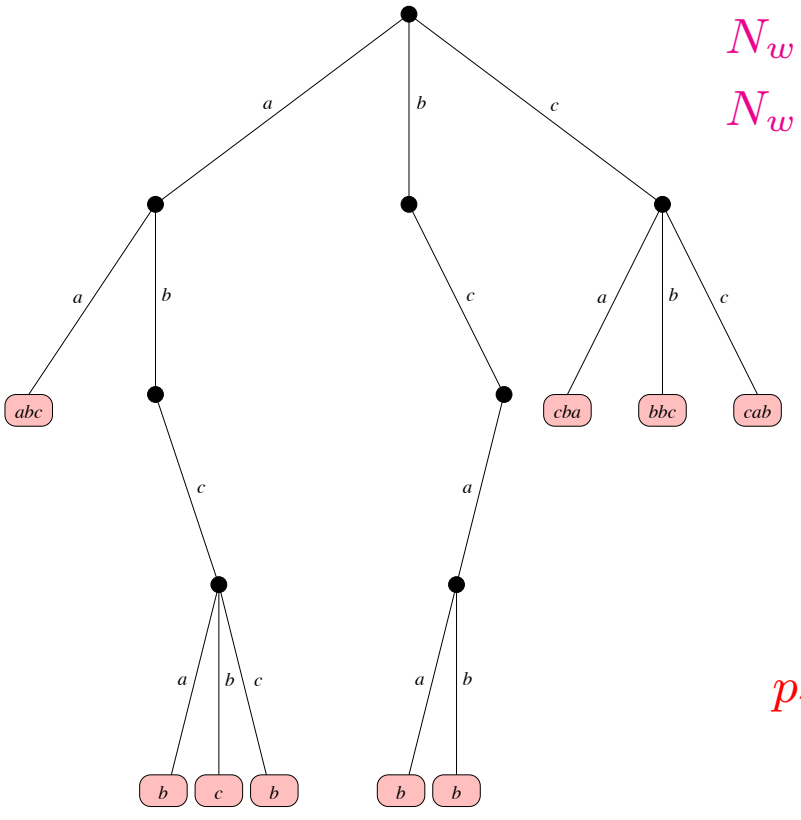

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The size, and the path length of a trie equal

$$R = \sum_{w \in \Sigma^\star} \mathbf{1}_{[N_w \geq 2]} \qquad T = \sum_{w \in \Sigma^\star} \mathbf{1}_{[N_w \geq 2]} \cdot N_w,$$

Central role of

p_w :=the probability that a word begins with prefix w.

A realistic framework for sorting or searching.

Keys are viewed as words and are compared [wrt the lexicographic order].

The realistic unit cost is now the symbol–comparison.

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$$A = a_1\, a_2\, a_3 \ldots a_i \ldots \qquad \text{and} \qquad B = b_1\, b_2\, b_3 \ldots b_i \ldots$$

equals $k+1$, where k is the length of their largest common prefix

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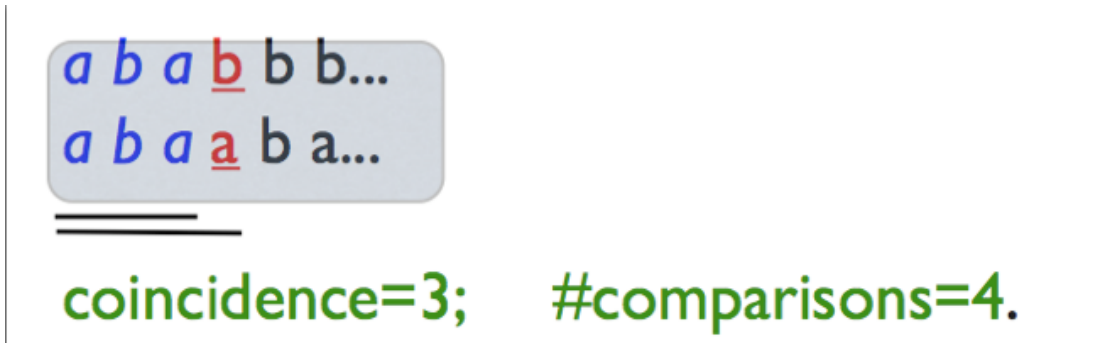

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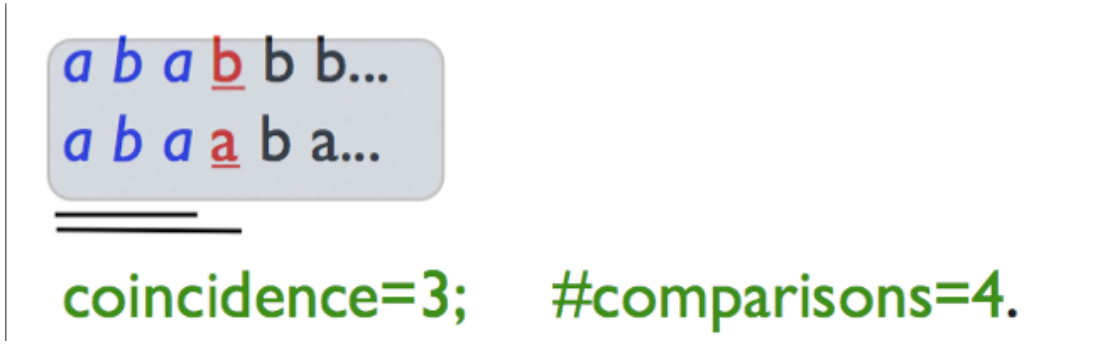

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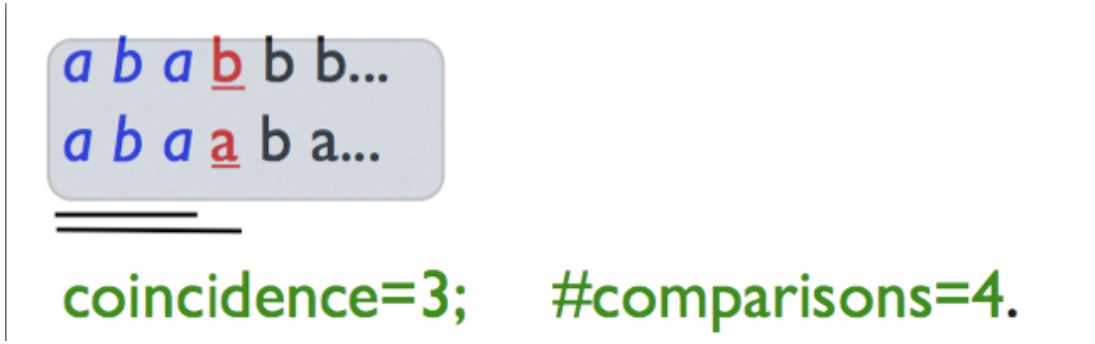

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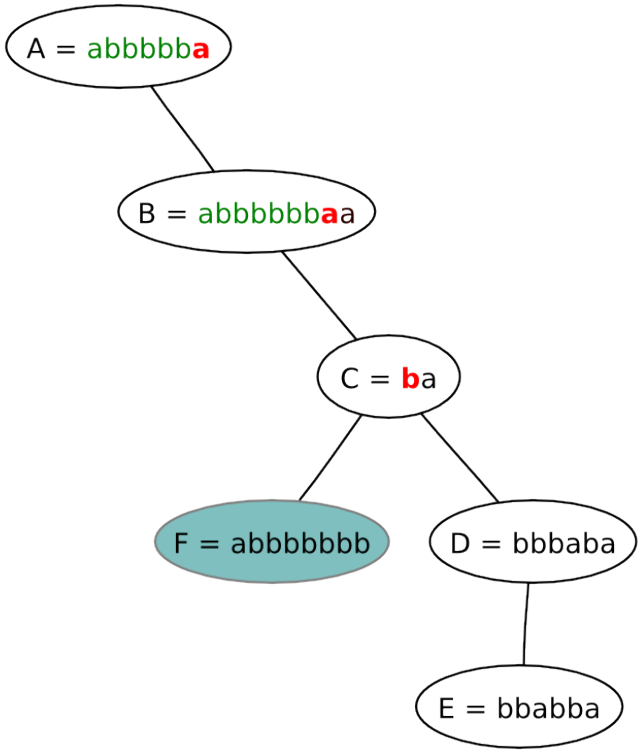
The example of the binary search tree (BST)



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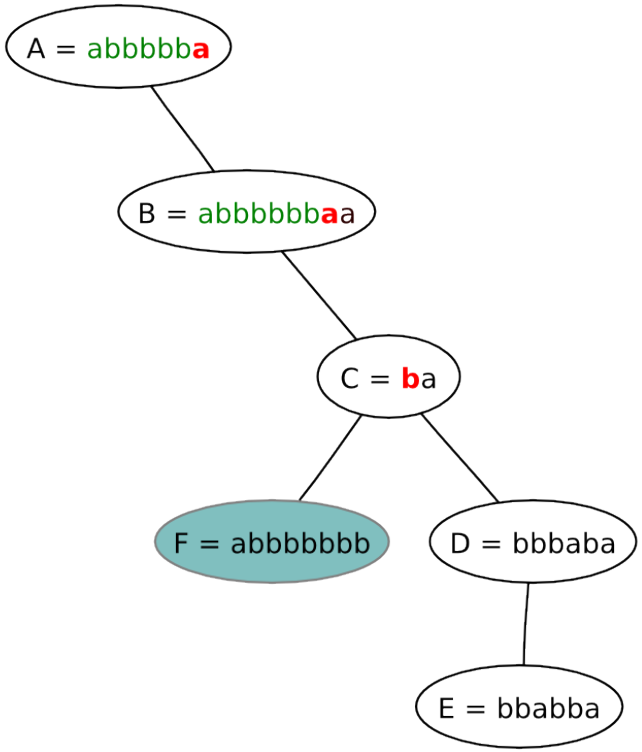

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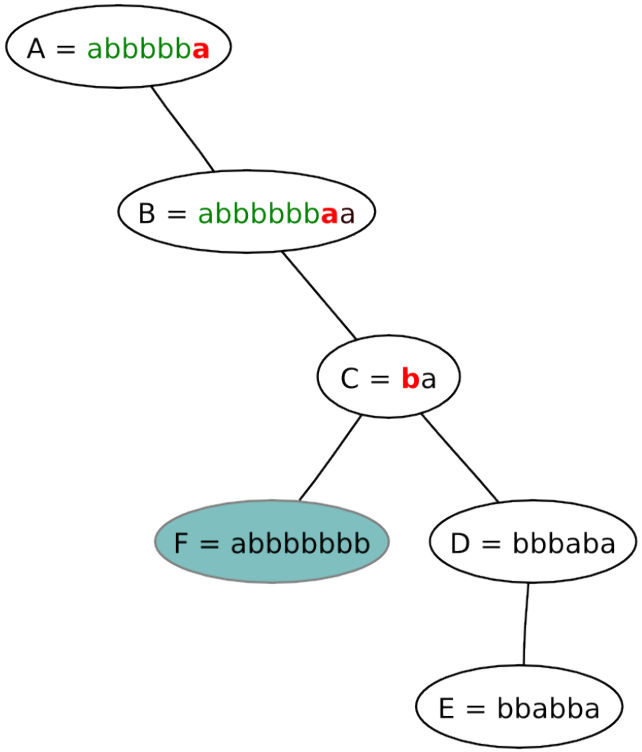

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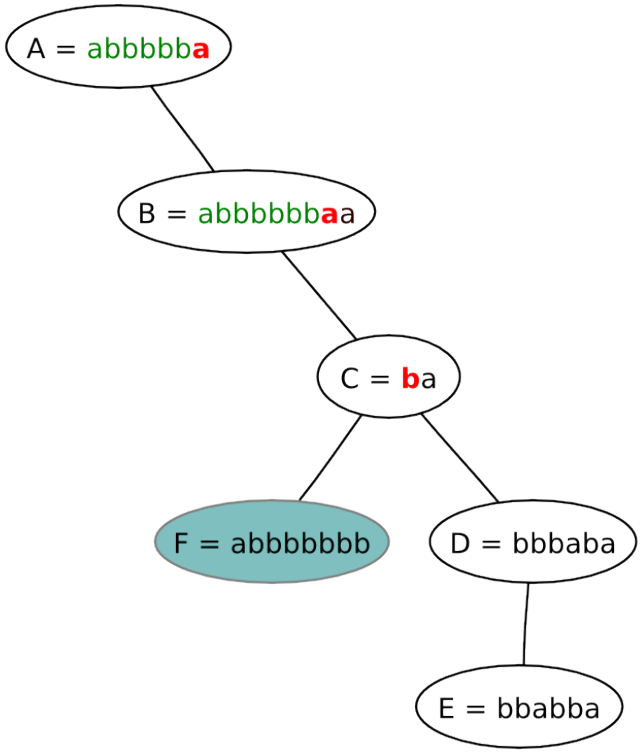

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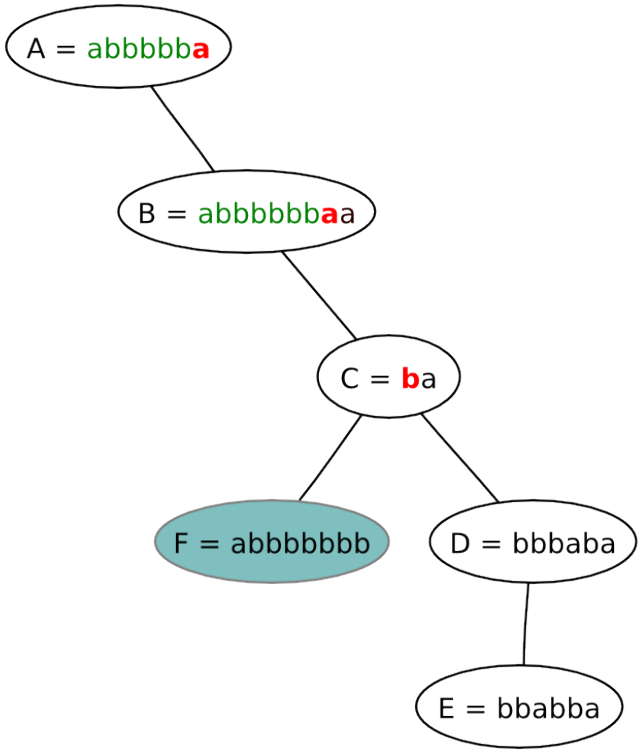

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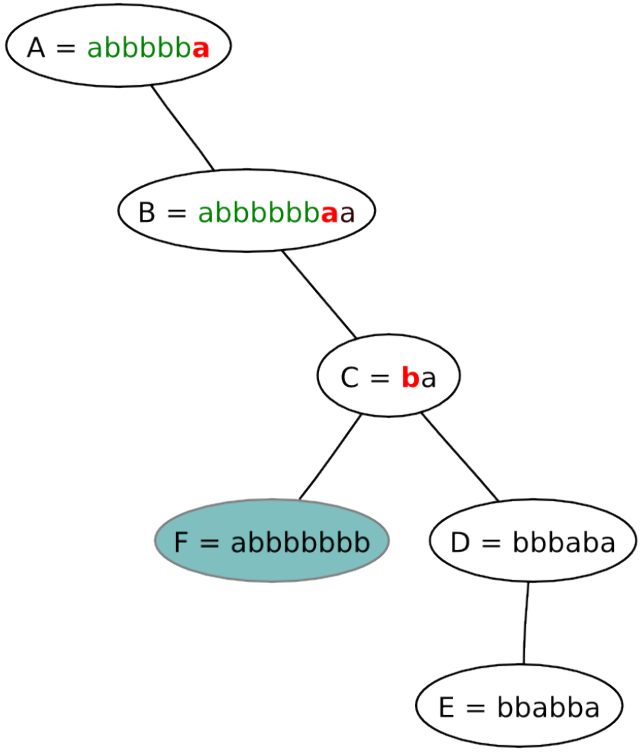

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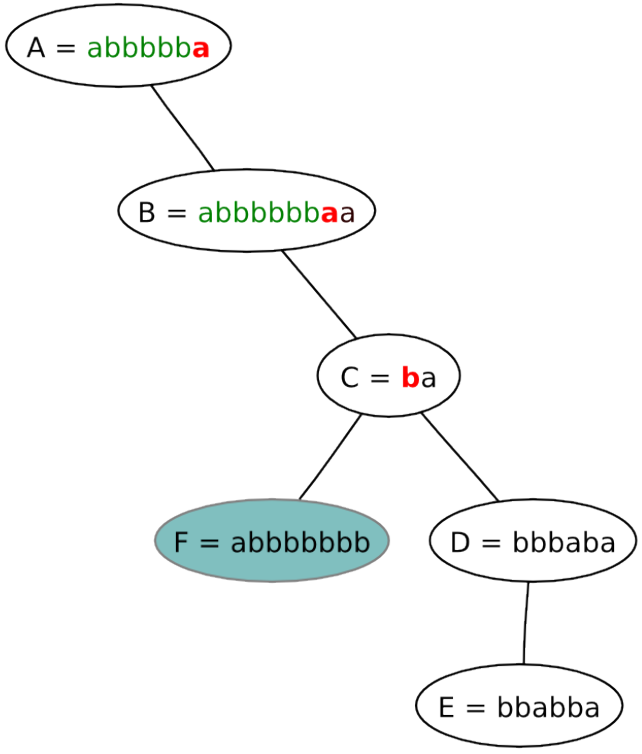

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This defines the symbol-path-length of a BST based on the coincidence
We perform a probabilistic study of this symbol path-length

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For $w \in \Sigma^\star$, $p_w :=$ probability that a word begins with the prefix w.
The set $\{p_w, \quad w \in \Sigma^\star\}$ defines the source $\mathcal{S}$.

Fundamental role of the Dirichlet generating functions of the source

$$\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s, \qquad \Lambda_k(s) = \sum_{w \in \Sigma^k} p_w^s, \qquad \left[\Lambda = \sum_{k \geq 0} \Lambda_k \right]$$

Remark: $\Lambda_k(1) = 1$ for any k, $\qquad \Lambda(1) = \infty$.

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– they intervene in probabilistic analysis of algorithms and data structures.

Exact average-case analysis for Tries or BST's

$S_n^{(X)}$:= the mean path-length for the Trie $[X = T]$
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$$S_n^{(X)} = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi_X(k)$$

which involves a series ϖ_X at integer values k.

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This series $\varpi_X(s)$ is closely related to the Dirichlet series of the source

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Nice exact formulae, not easy to deal with, due to the alternating signs

Asymptotic analysis.

The residue formula transforms the sum into an integral with $1 < d < 2$.

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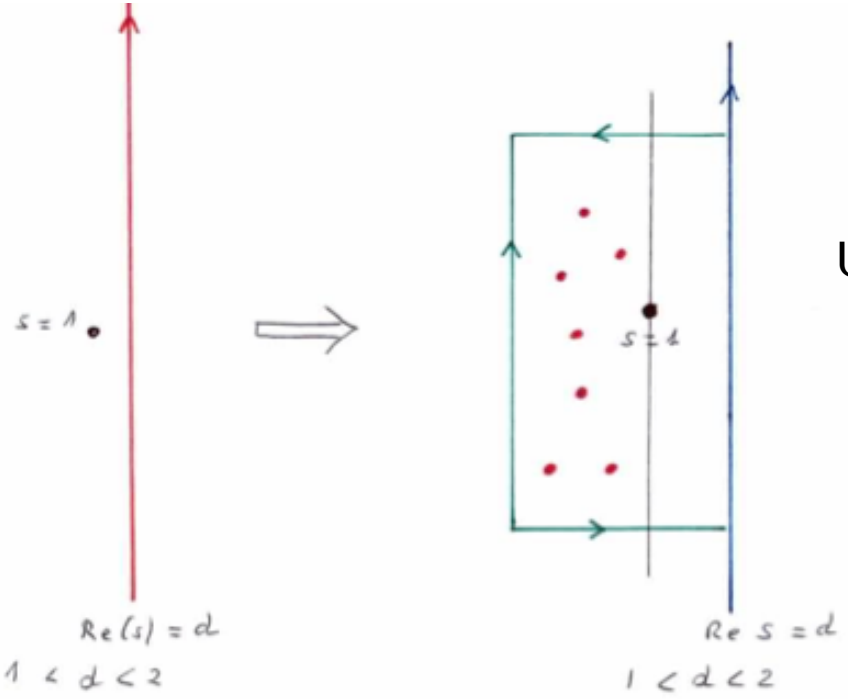

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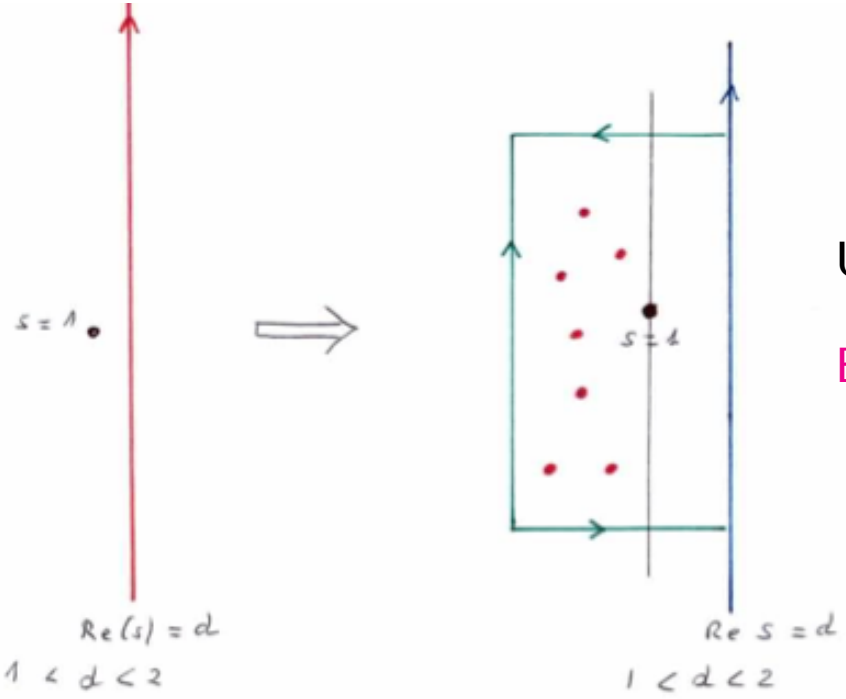

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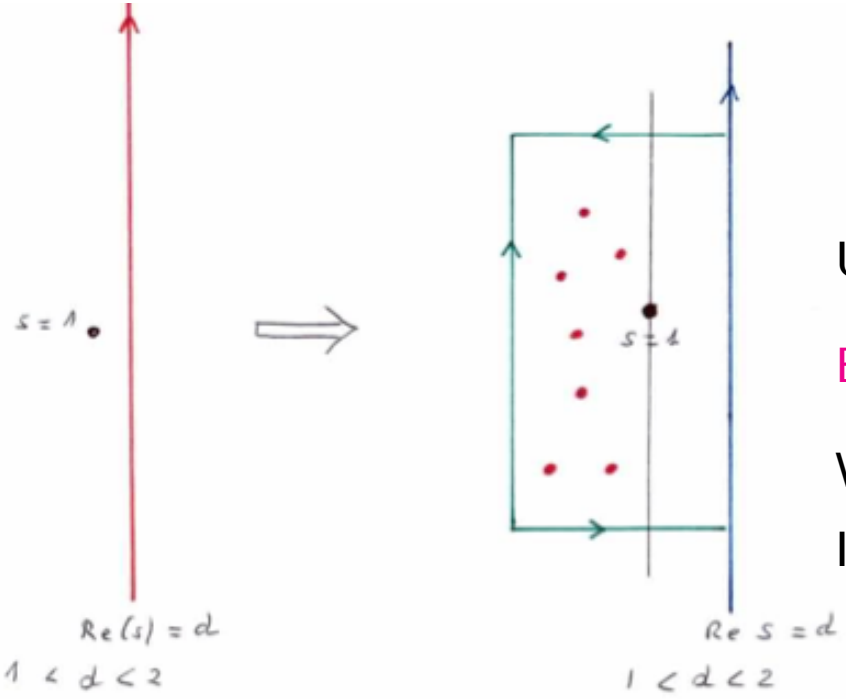

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Where are the red singularities closest to $\Re s = 1$?

Is $\Lambda(s)$ of polynomial growth on the green contour?

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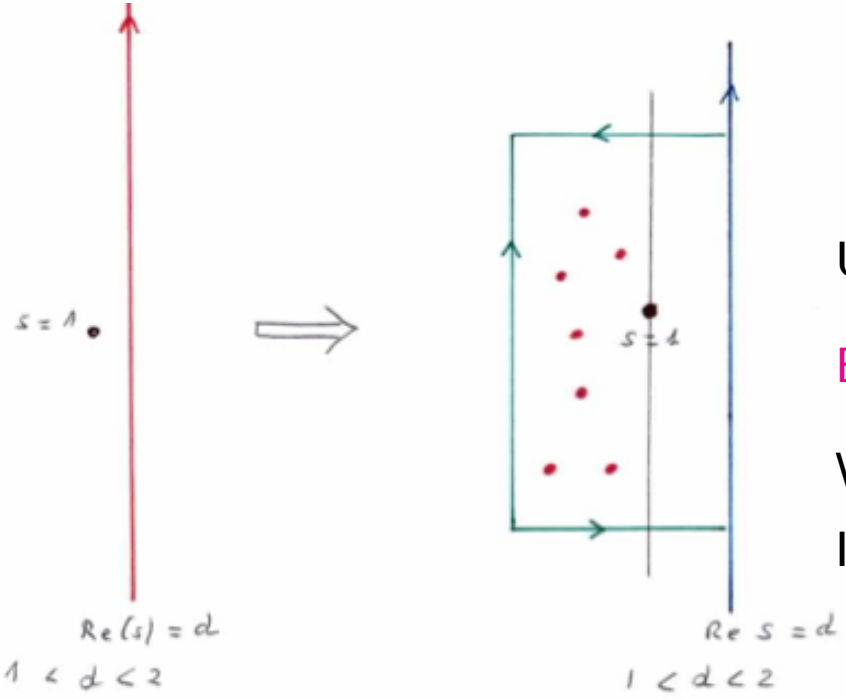

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Importance of the existence of a region $\mathcal{R}$

– which contains only $s = 1$ as a pole – where $\Lambda(s)$ is of polynomial growth.

Tameness of the source

Main results

[Clément, Flajolet, V. (2001), Clément, Flajolet, Fill, V. (2009)]

Consider n words independently drawn from the same tame source. Then:

The mean path-length T_n of the Trie satisfies	The mean symbol path-length B_n of the BST satisfies
$T_n \sim \frac{1}{h_{\mathcal{S}}}\, n \log n.$	$B_n \sim \frac{1}{h_{\mathcal{S}}}\, n \log^2 n.$

Here, $h_{\mathcal{S}}$ is the entropy $h_{\mathcal{S}}$ of the source $\mathcal{S}$, defined as

$$h_{\mathcal{S}} := \lim_{k\to\infty} \left[\frac{-1}{k} \sum_{w\in\Sigma^k} p_w \log p_w \right],$$

where p_w is the probability that a word begins with prefix w.

Plan of the talk.

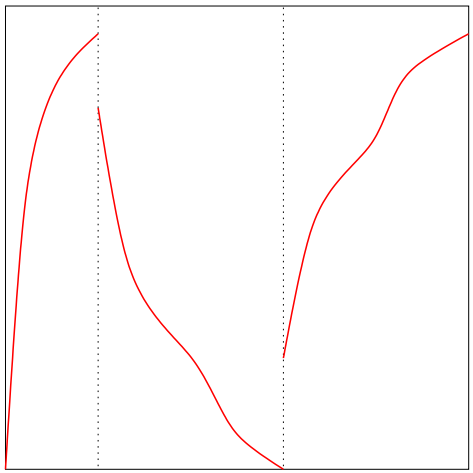
– General motivations: Dirichlet generating functions and tameness

– An important class of "natural" sources: dynamical sources
= sources associated to dynamical systems

– Tameness in the case of dynamical sources

– Conclusion and possible extensions.

A dynamical source = a source built with a dynamical system [V. 1998]



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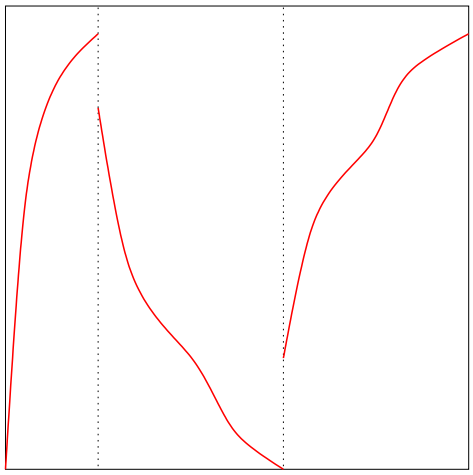


A dynamical system $(\mathcal{I}, T)$ is defined by

– an alphabet Σ denumerable (possibly infinite),

– a topological partition of $\mathcal{I} :=]0,1[$
with open intervals $\mathcal{I}_{m, m\in\Sigma}$,

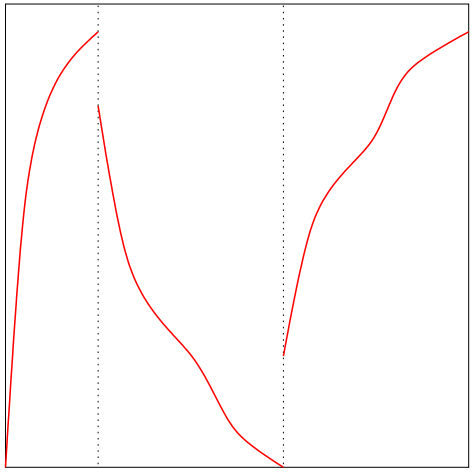
– an encoding mapping σ equal to m on each $\mathcal{I}_m$,

– a shift mapping T

– each $T|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ on $\mathcal{I}_m$

– The local inverse of $T|_{\mathcal{I}_m}$ is denoted by h_m.

A dynamical source = a source built with a dynamical system [V. 1998]



A dynamical system $(\mathcal{I}, T)$ is defined by

– an alphabet Σ denumerable (possibly infinite),

– a topological partition of $\mathcal{I} :=]0,1[$

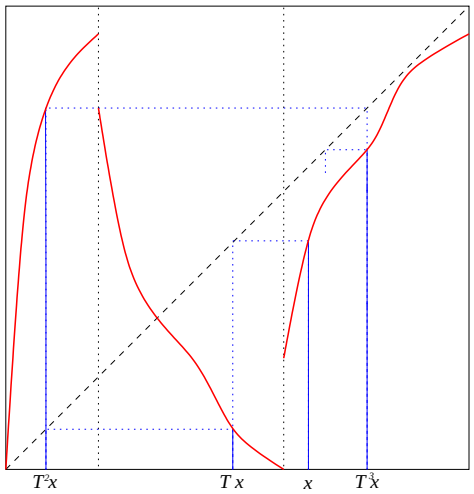
with open intervals $\mathcal{I}_{m, m\in\Sigma}$,

– an encoding mapping σ equal to m on each $\mathcal{I}_m$,

– a shift mapping T

– each $T|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ on $\mathcal{I}_m$

– The local inverse of $T|_{\mathcal{I}_m}$ is denoted by h_m.



$M(x) = (c, b, a, c \ldots)$

This gives rise to a source:

On an input x of $\mathcal{I}$, it outputs the word

$$M(x) := (\sigma x, \sigma T x, \sigma T^2 x, \ldots).$$

When an initial density is chosen on $\mathcal{I}$,

this induces (via M) a probabilistic model on Σ^∞

= a dynamical source

Strong relations between the geometry of the system,

the correlations between symbols and the probabilistic properties of the source.

Two geometric characteristics of the system:

- The position of the branches $T(\mathcal{I}_k)$ w.r.t $\mathcal{I}_m$
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Particular cases: simple sources and affine branches

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:= a complete system with affine branches and uniform initial density

A Markov chain

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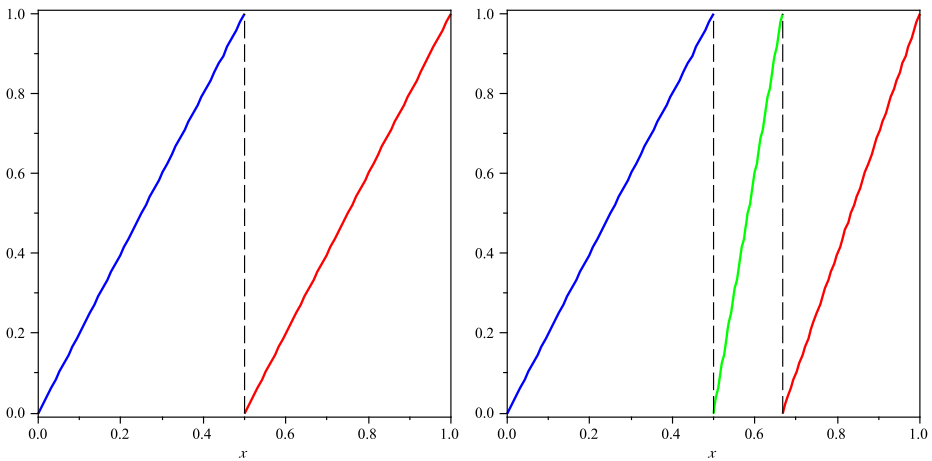

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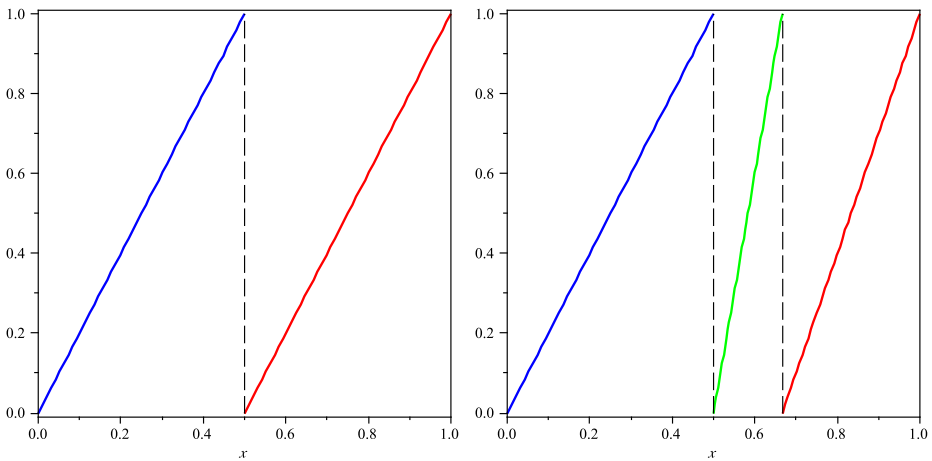

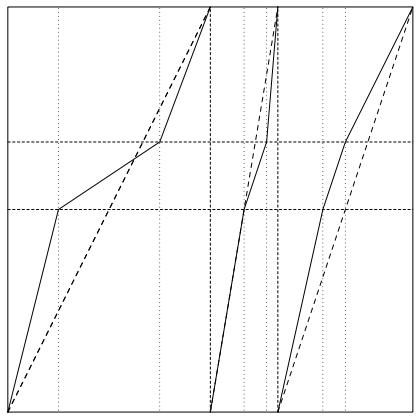

General case of interest = the Good Class gathers

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– with expansive branches : $|T'(x)| \geq \rho > 1$.

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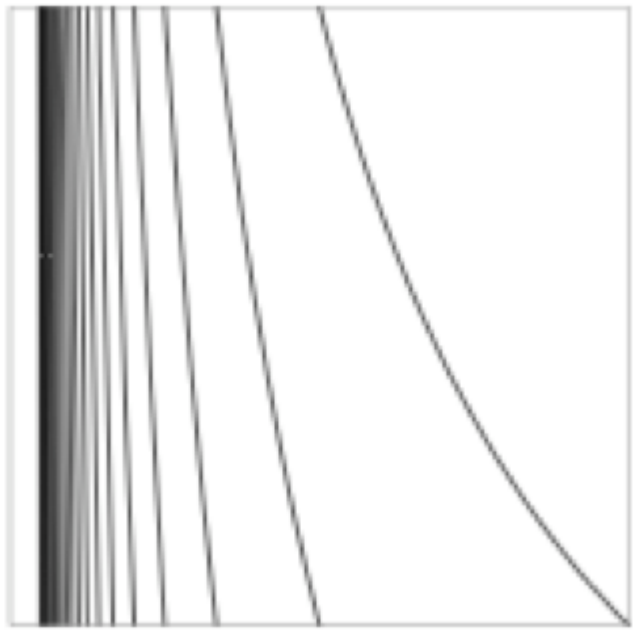
Main instance: the Euclidean source defined with $T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$



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Markov chains, defined by – the vector R of initial probabilities (r_i)

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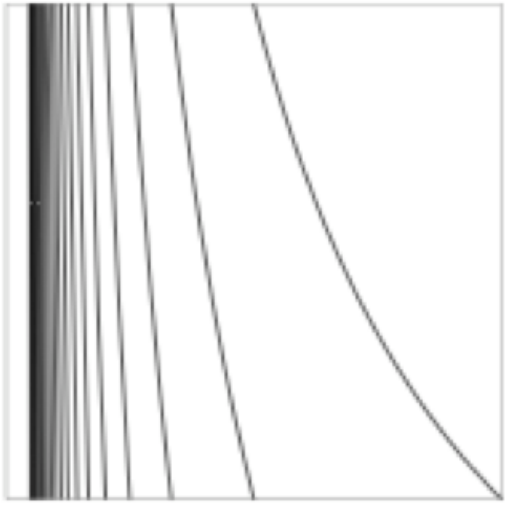
A general dynamical source

$\Lambda(s)$ closely related to $(I - \mathbb{H}_s)^{-1}$

where $\mathbb{H}_s$ is the (secant) transfer operator of the dynamical system.

The density transformer and the transfer operators



The operator $\mathbf{H} := \sum_{m \in \Sigma} \mathbf{H}_{[m]}$

with $\mathbf{H}_{[m]}[f](x) = |h'_m(x)| \cdot f \circ h_m(x)$

is the density transformer of the dynamical system.

It describes the evolution of the density.

For a density f on $[0, 1]$,

$\mathbf{H}[f]$ is the density on $[0, 1]$ after one iteration.

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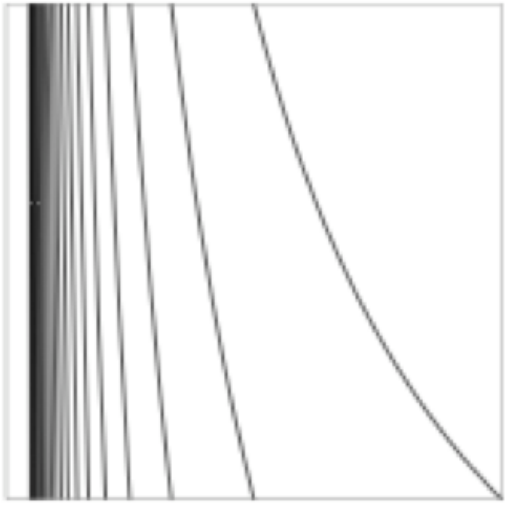

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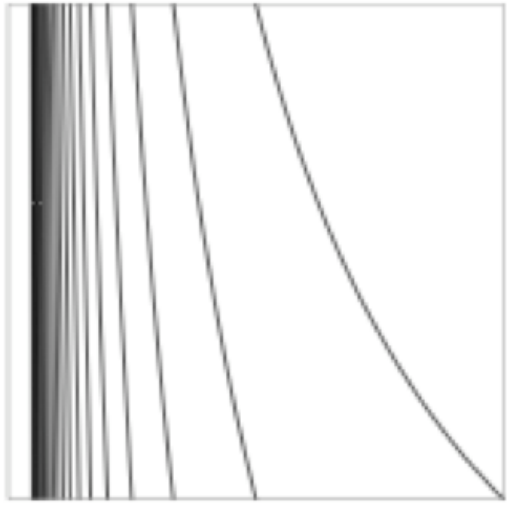

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Transfer operator (Vallée, 2000) [secant version]

$$\mathbb{H}_s := \sum_{m\in\Sigma} \mathbb{H}_{s,[m]} \quad \text{with} \quad \mathbb{H}_{s,[m]}[F](x,y) = \left|\frac{h_m(x) - h_m(y)}{x-y}\right|^s F(h_m(x), h_m(y))$$

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For $s = 1$, $\mathbb{H}_1$ is an extension of $\mathbf{H}$ and has an eigenvalue equal to 1.
For a system of the Good Class, $s \mapsto \Lambda(s)$ has a simple pole at $s = 1$

Plan of the talk.

– General motivations: Dirichlet generating functions and tameness

– An important class of sources: dynamical sources.

– Tameness of dynamical sources

– Conclusion and possible extensions.

What happens on the left of the vertical line $\Re s = 1$?

It is important for the analysis to deal with a region $\mathcal{R}$ where $\Lambda(s)$ is tame

– it is analytic (except for $s = 1$) and of polynomial growth ($\Im s \to \infty$)

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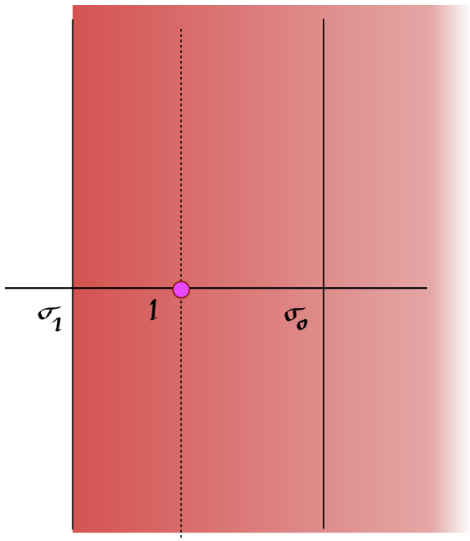
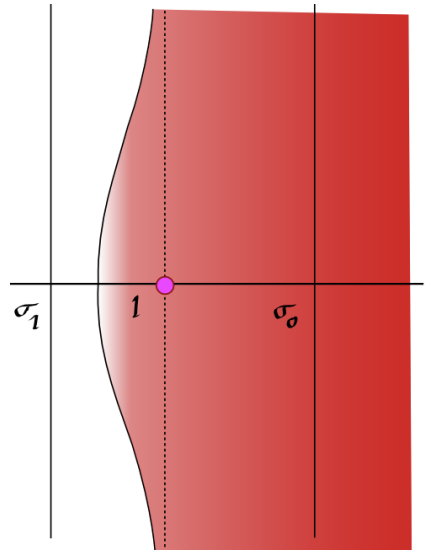


Situation 1

Vertical strip

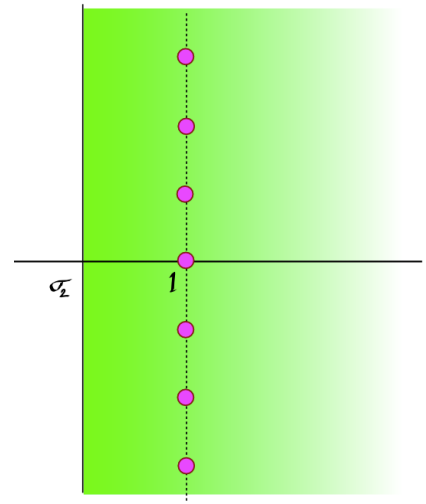
$1 - \sigma \leq a$



Situation 2

Hyperbolic region

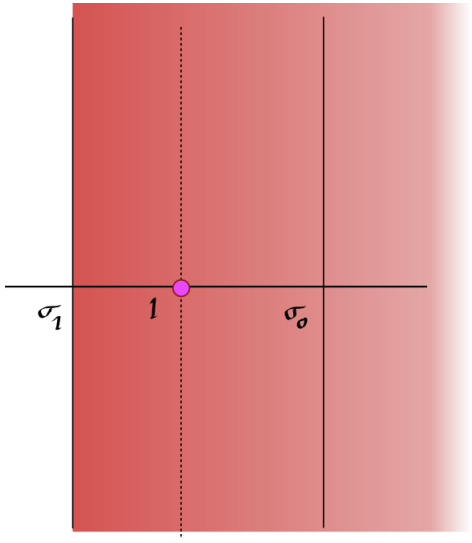
$1 - \sigma \leq t^{-\alpha}$



Situation 3

Vertical strip with holes

Different possible regions on the left of $\Re s = 1$ where $\Lambda(s)$ is tame.



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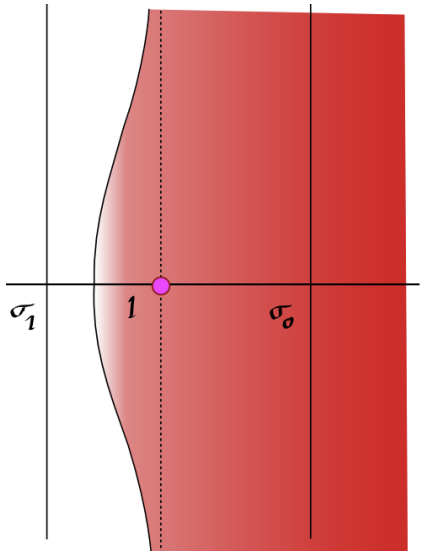
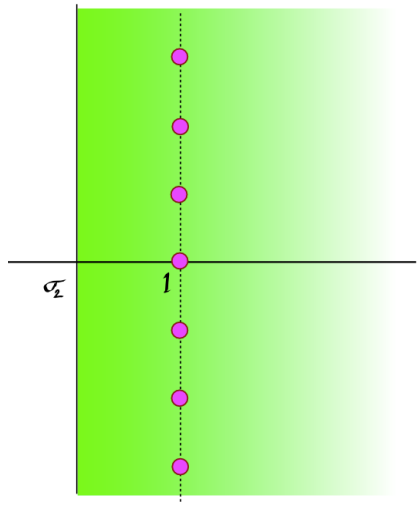


Situation 2

Hyperbolic region



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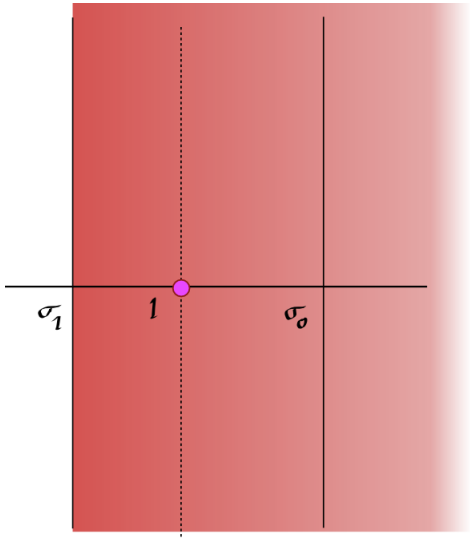

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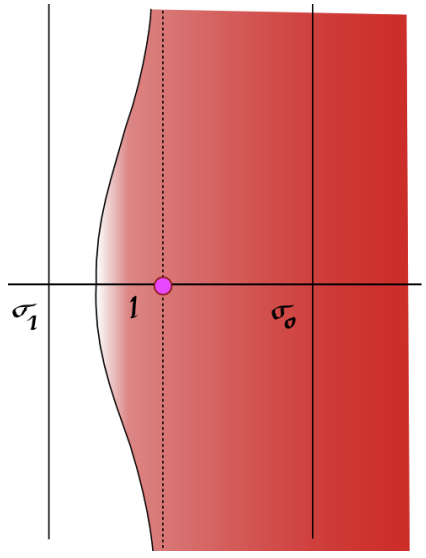
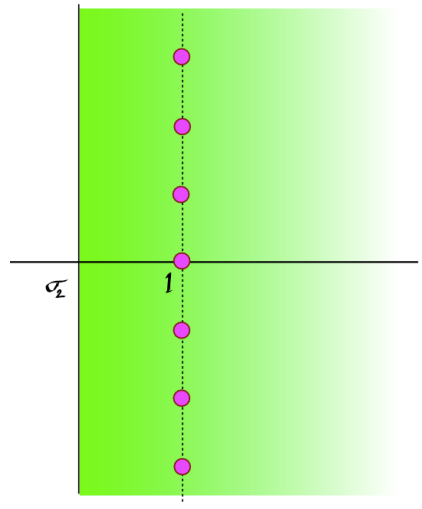


Situation 2

Hyperbolic region



Situation 3

Vertical strip with holes

For which simple sources do these different situations occur?

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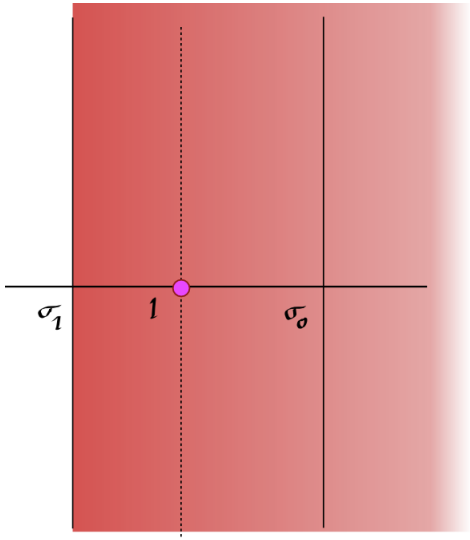

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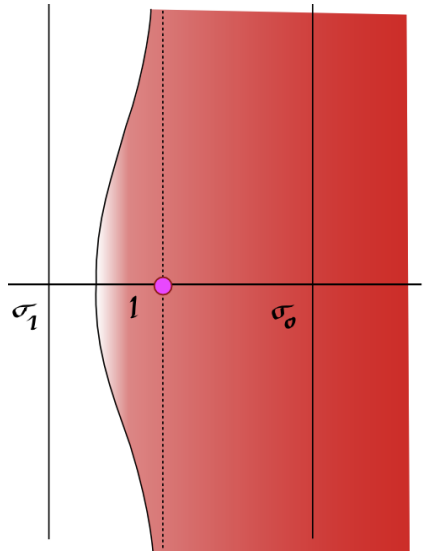
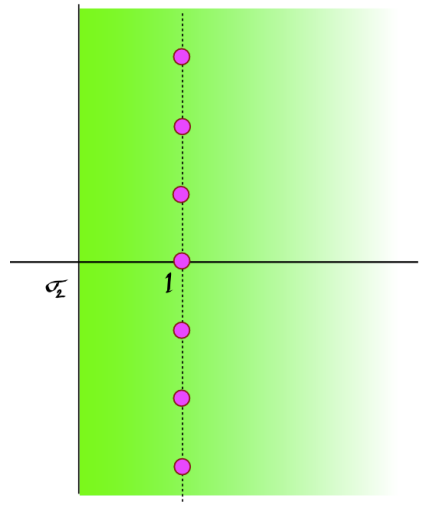


Situation 2
Hyperbolic region



Situation 3
Vertical strip with holes

For which simple sources do these different situations occur?

For memoryless sources relative to probabilities $(p_1, p_2, \ldots, p_r)$

– S1 is impossible

– S3 occurs when all the ratios $\log p_i / \log p_j$ are rational

– S2 occurs if there exists a ratio $\log p_i / \log p_j$
which is "diophantine" [badly approximable by rationals]

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(i) $\mathcal{Z} \neq \emptyset \iff \log p_2/\log p_1$ is rational

(ii) If $\mathcal{Z} = \emptyset$, then the poles of $\Lambda(s)$ close to $\Re s = 1$
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The irrationality exponent $\mu(x)$ of a number x equals μ if, for any $\nu > \mu$,
the set of pairs $(a,b) \in \mathbb{Z}^2$ for which $\left|x - \dfrac{a}{b}\right| \leq \dfrac{1}{b^\nu}$ is finite

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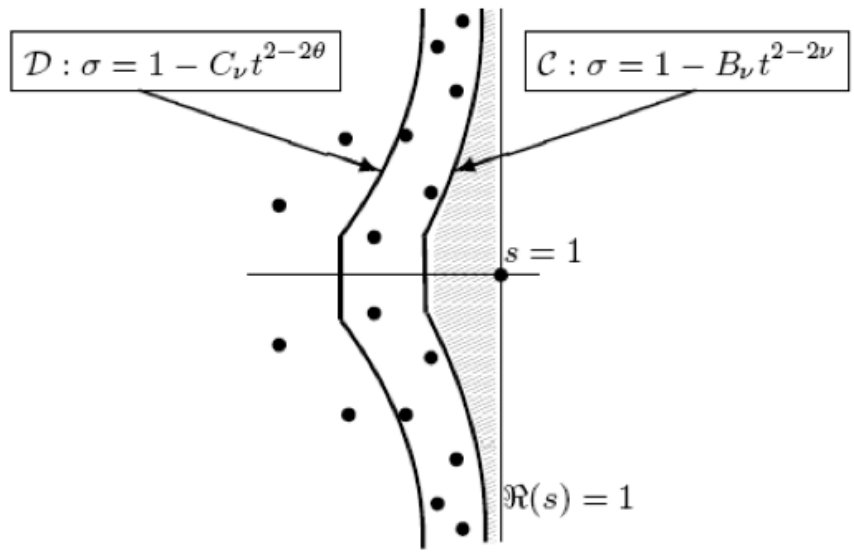
The shape of the tameness region
is related to $\mu(\log p_2/\log p_1)$.

If $\mu(\log p_2/\log p_1) = \mu$
then, for any θ, ν with $\theta < \mu < \nu$,
the tameness region is as shown:
[Flajolet-Roux-V. 2010]



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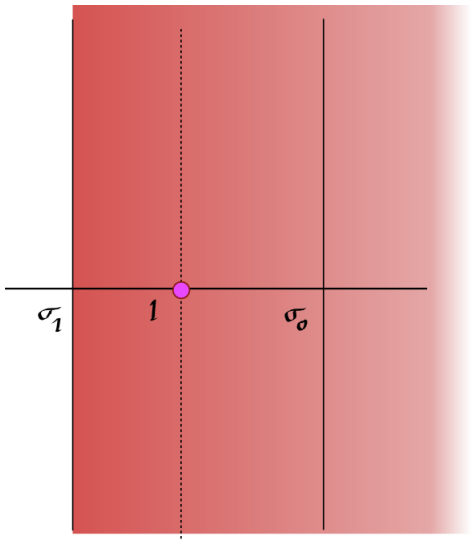
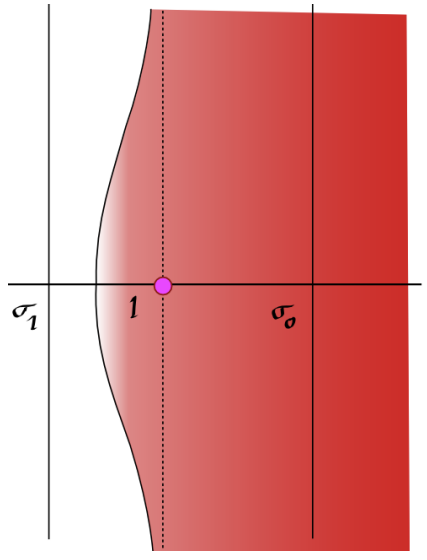


Situation 1

Vertical strip

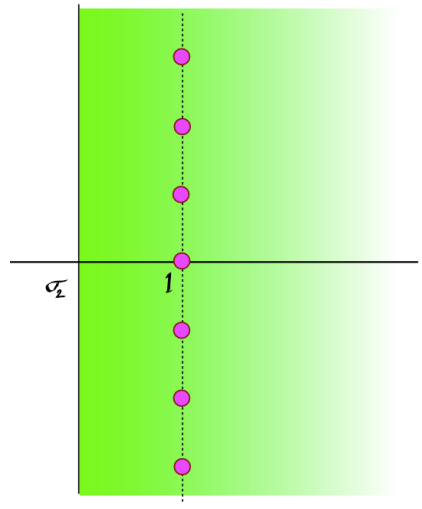
Geometric condition



Situation 2

Hyperbolic region

Arithmetic condition



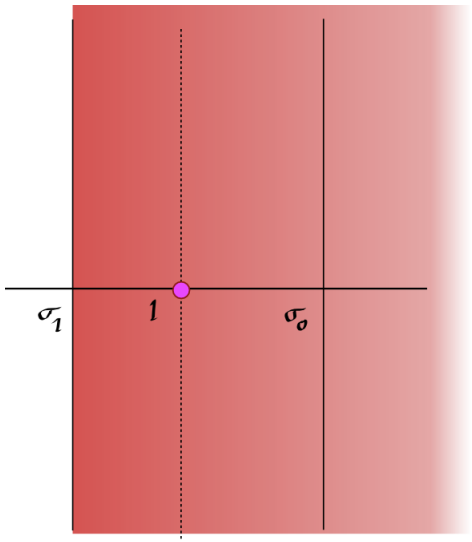
Situation 3

Vertical strip with holes

Periodicity condition

For which general dynamical sources do these different situations occur?

Different possible regions on the left of $\Re s = 1$ where $\Lambda(s)$ is tame.



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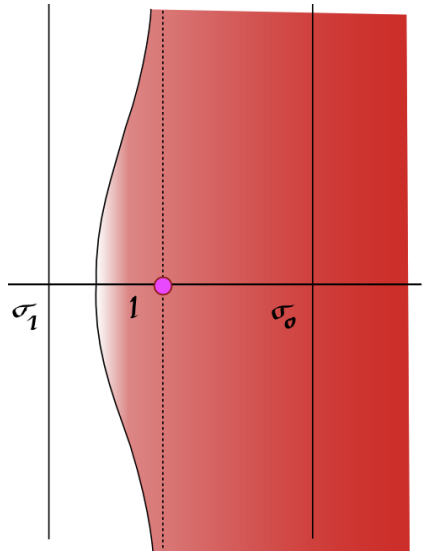
Vertical strip

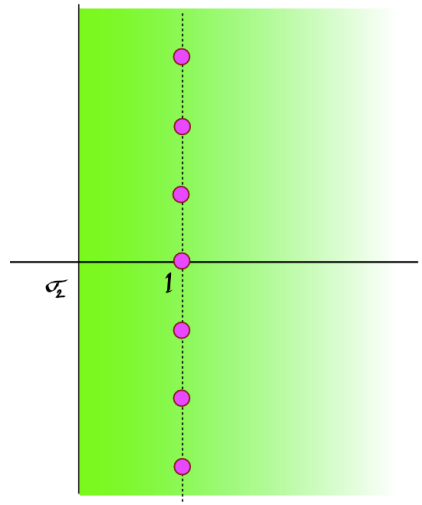
Geometric condition



Situation 2

Hyperbolic region

Arithmetic condition



Situation 3

Vertical strip with holes

Periodicity condition

For which general dynamical sources do these different situations occur?

– S1 occurs when "the branches are not too often of the same shape".

– S3 occurs only if the source is conjugated to a simple source.

– S2 occurs if a extension of the following condition holds:

"there exists a ratio $\log p_i / \log p_j$ which is "diophantine"

Situation 1- Existence of a vertical strip where $\Lambda(s)$ is tame

The condition UNI expresses that

"the branches of the dynamical system are not too often of the same shape"

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Dolgopyat (98) proves the result for the plain transfer operator, in the case of a finite number of branches

– Baladi and V. (03) extend the result for an infinite number of branches

– Cesaratto and V. (09) extend the result to the secant transfer operator.

Situation 2- Existence of a hyperbolic region where $\Lambda(s)$ is tame

The condition DIOP extends the arithmetic condition

"There exists a ratio $\log p_i / \log p_j$ which is diophantine"

For a complete system, each branch h has a fixed point denoted by $h^\star$.

The derivatives $|h'(h^\star)|$ replace the probabilities of the memoryless case.

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Dolgopyat (98) proves the result for the plain transfer operator, in the case of a finite number of branches – Roux and V. (2010) extend the result : for an infinite number of branches and for the secant transfer operator.

Plan of the talk.

– General motivations: Dirichlet generating functions and ttameness

– An important class of sources: dynamical sources.

– Tameness of dynamical sources

– Conclusion and possible extensions.

Conclusions.

Description of a framework which

– unifies the analyses for text algorithms and searching/sorting algorithms

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– provides sufficient conditions for tameness of dynamical sources

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Description of a framework which

– unifies the analyses for text algorithms and searching/sorting algorithms

– provides a general model for sources
– shows the importance of the Dirichlet generating functions
– explains the importance of tameness for sources

– defines a natural subclass of sources, the dynamical sources
– provides sufficient conditions for tameness of dynamical sources

– provides probabilistic analyses for algorithms built on tame sources.

Possible extensions and work in progress

I– Classification of sources

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– Place of dynamical sources amongst general sources:

– A dynamical source = limit of Markov chains with increasing order?

– Comparing dynamical sources with Markov chains of variable length

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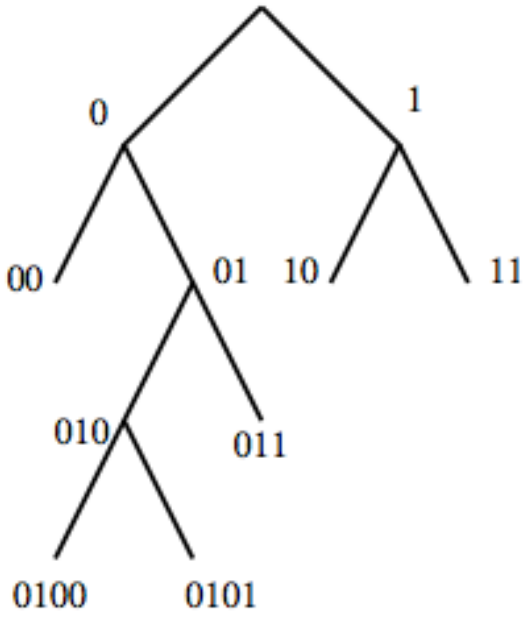

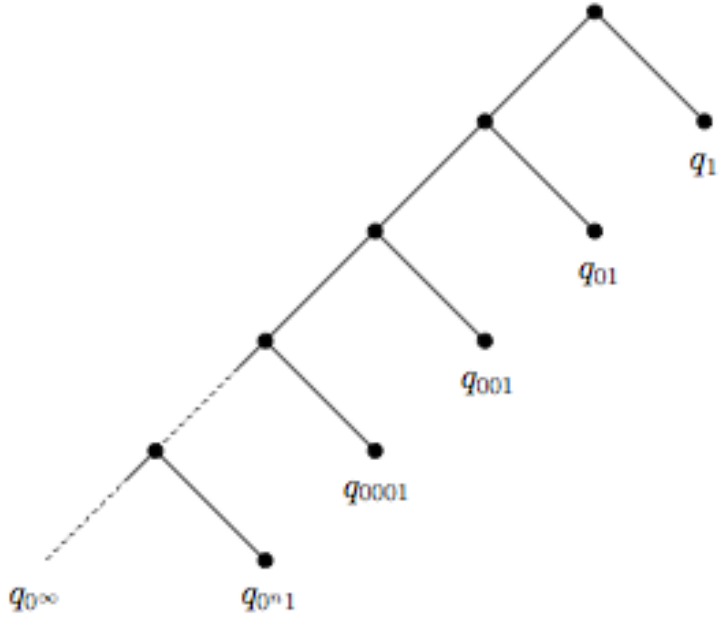

Possible extensions and work in progress

II– Realistic analyses of other algorithms and other structures

– Analysis of other sorting algorithms
  – Analysis of `Insertion` Sort easy
  – Analysis of `QuickSelect` already done
  – And `Selection` algorithm ?

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– Analysis of the DST structure?