Abelian returns in Sturmian words

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jointly with L. Q. Zamboni

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Σ^* – finite words over Σ
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- $w \in \Sigma^\omega$ is recurrent if each of its factors occurs infinitely many times in w.
- $F(w)$: the set of factors of a finite or infinite word w

return words

Definition

$w = w_1 w_2 \dots$ a recurrent infinite word, $u \in F(w)$,

let $n_1 < n_2 < \dots$ be all integers n_i such that $u = w_{n_i} \dots w_{n_i+|u|-1}$

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- introduced independently by F. Durand, C. Holton and L. Q. Zamboni, 1998, and used for a characterization of primitive substitutive sequences
- and then for different problems of combinatorics on words, symbolic dynamical systems and number theory (L. Vuillon, J. Justin, J.-P. Allouche, J. D. Davinson, M. Queffélec, I. Fagnot, J. Cassaigne...)

Characterization of Sturmian words via return words

An infinite word has k returns, if each of its factors has k returns.

A characterization of Sturmian words via return words:

Theorem (L. Vuillon, J. Justin, 2000–2001)

A recurrent infinite word has two returns if and only if it is Sturmian.

Characterization of periodicity via return words

Characterization of periodicity via return words:

Proposition (L. Vuillon, 2001)

A recurrent infinite word is ultimately periodic if and only if there exists a factor having exactly one return word.

Abelian returns

$u \in \Sigma^*$, $a \in \Sigma$, $|u|_a$ – the number of occurrences of the letter a in u

$u, v \in \Sigma^*$ are abelian equivalent if $|u|_a = |v|_a$ for all $a \in \Sigma$

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u has k abelian returns in w, if the set of abelian returns of u consists of k abelian classes

I. e., we take

- factors up to abelian equivalence
- return words up to abelian equivalence

Example

the Thue-Morse word

$$t = 0110100110010110\ldots$$

Consider abelian returns of its factor 01:

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Consider abelian returns of its factor 01:

01 → 0 — ab. ret. 0

01 → 10 — ab. ret. 01

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01 → 0 ab. ret. 0
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symmetrically

10 → 0 ab. ret. 1
10 → 01 ab. ret. 10

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01	0	ab. ret. 0
	10	ab. ret. 01

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	01	ab. ret. 10

three abelian returns: 0, 1 and $01 \approx^{ab} 10$.

Main result

A characterization of Sturmian words via the abelian returns:

Theorem

An aperiodic recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian returns.

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A characterization of Sturmian words via the abelian returns:

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Remind a characterization by Vuillon:
Sturmian $\Leftrightarrow$ each factor has two (normal) returns

Idea of proof

Sturmian $\Rightarrow$ two or three returns

Based on a characterization of balanced words via orderings by O. Jenkinson, L. Q. Zamboni (2004)

- $w \in \{0,1\}^q$ balanced word with $|w|_1 = p$, $\gcd(p,q) = 1$.

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  $\sigma(w_0 \ldots w_{q-1}) = w_1 \ldots w_{q-1} w_0$.

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- the lexicographic ordering of $\{\sigma^i(w) : 0 \le i < q\}$:
  $w_{(0)} <_L w_{(1)} <_L \cdots <_L w_{(q-1)}$

  Lexicographic array $A[w]$: $q \times q$ matrix whose ith row is $w_{(i)}$

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- j-th column of A is $\sigma^{jp} u$, where $u = 0^{q-p} 1^p$

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The lexicographic array:

$$A[w] = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Idea of proof

two or three returns $\Rightarrow$ Sturmian

considering abelian returns to factors of special type and restricting the form of words (quite technical)

Structure of abelian returns in Sturmian words

Properties of abelian returns in Sturmian words

- Abelian returns of factors of a Sturmian word are either letters or of the form aBb, where $a \neq b$ are letters, and B is a bispecial factor.

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  A factor of a Sturmian word is called singular if it is the only factor in its abelian class.

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- If a factor of a Sturmian word has three abelian returns of lengths $l_1 \leq l_2 < l_3$, then $l_3 = l_1 + l_2$

A simple sufficient condition for periodicity via abelian returns:

Lemma

Let $|\Sigma| = k$. If each factor of a recurrent infinite word over the alphabet Σ has at most k abelian returns, then the word is periodic.

Remark: not necessary condition for periodicity!

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Example: an infinite aperiodic word in $\{110010, 110100\}^\omega$
the factor 11 has one abelian return $110010 \approx^{ab} 110100$

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periodicity $\not\Rightarrow$ $\exists$ factor having one abelian return
Example: $w = (00110100101100110011 0011)^\omega$

Returns to abelian classes

Another version:

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w infinite recurrent word, $u \in F(w)$

w has k returns to the abelian class of u, if the set of abelian returns of u consists of k different words.

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Remind: in the sense of previous definition 01 has 3 abelian returns, because $01 \approx^{ab} 10$.

Characterization 2

In the sense of the second definition the characterization also holds:

Theorem

An aperiodic recurrent infinite word w is Sturmian if and only if for each $u \in F(w)$ the word w has two or three returns to the abelian class of u.

The end

Thank you!