Pattern Avoidance and HDOL Words

Pascal Ochem

LIRMM, CNRS, Montpellier

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Patterns

Pattern: finite word over $\{A, B, C, \cdots\}$.
Word: over $\Sigma_k = \{0, 1, \cdots, k-1\}$.
Occurrence: obtained from the pattern by replacing each letter by a non-empty word.

Example

- 10111011 is an occurrence of ABCACB with $A = 10$, $B = 1$, $C = 1$.
- 00101001 contains 3 occurrences of ABBA: 001010, 010100 and 1001.

w avoids P if w contains no occurrence of P as a factor.
Avoidability index: $\lambda(P)$ is the smallest k such that there exists an infinite word over Σ_k avoiding P.

Avoidability index (1)

- $\lambda(AA) = 3$: infinite ternary square-free words.
- $\lambda(AAA) = 2$: infinite binary cube-free words.

Remark: $\lambda(P) \geq 2$

Divisibility:
ABCABCAAC contains an occurrence *AABB*
$\Longrightarrow \lambda(ABCABCAAC) \leq \lambda(AABB)$
$\Longrightarrow$ partial order among patterns

Avoidability index (2)

- $\cdots \leq \lambda(ABCABC) \leq \lambda(ABAB) \leq \lambda(AA) = 3$.
- $\cdots \leq \lambda(AABBCC) \leq \lambda(AABB) \leq \lambda(AA) = 3$.

We have:

- $\lambda(ABCABC) = 2$, $\lambda(ABAB) = 3$.
- $\lambda(AABBCC) = 2$, $\lambda(AABB) = 3$.

Bounds on the avoidability index

Lower bounds: To prove that $\lambda(P) \geq k$, check that all words over Σ_{k-1} are finite by lexicographic enumeration. This is the easy part.

Upper bounds: To prove that $\lambda(P) \leq k$, construct an infinite word over Σ_k and prove that it avoids P.

Square-free words and square-free morphisms

$\lambda(AA) = 3$: infinite ternary square-free word.

square-free morphisms:

- $0 \mapsto 01201$, $1 \mapsto 020121$, $2 \mapsto 0212021$.
- $0 \mapsto 012021201210212021 0$,
  $1 \mapsto 120102012021020102 1$,
  $2 \mapsto 201210120102101210 2$.

More than 1.3^n ternary square-free words of size n.

Morphic words

Pure morphic word (DOL):
fixed point $m^\infty(a)$ of a morphism m such that $m(a) = aw$.
Morphic word (HDOL):
image $h(m^\infty(a))$ of a pure morphic word by a morphism h.

- $\lambda(AA) = 3$, avoided by pure morphic words.
- s: $0 \mapsto 0101$, $1 \mapsto 0011$, $2 \mapsto 1000$.
  The image by s of a square-free word avoids $ABCABC$.
- $\lambda(ABCABC) = 2$, avoided by a morphic word, but no pure morphic word.
- Exponentially many binary words avoid $ABCABC$.

Julien Cassaigne's conjecture:
$\lambda(P) = k$ if and only if there exists a morphic word over Σ_k avoiding P.

A square-free morphic word

m_1: $0 \mapsto 012$, $1 \mapsto 02$, $2 \mapsto 1$.

$m_1^\infty(0)$ is the only infinite ternary word avoiding

- squares, 010, and 212.
- squares and $0u1u0$ for $u \in \Sigma_3^*$.

m_1 is not square-free:
$m_1(0u1u0) = 012m_1(u)02m_1(u)012$ contains $(2m_1(u)0)^2$.

Characterization

- A set S of forbidden patterns and factors.
- A morphic word w over Σ_k.

S characterizes w if w avoids S and for every finite factor f of w, $S \cup f$ is unavoidable over Σ_k.

Example

- $\{AA, 010, 212\}$ characterizes $m_1^\infty(0)$.
- $\{ABABA, 000, 111\}$ characterizes the Thue-Morse word.

This notion is defined to handle extendability issues.
Makes no difference between a pattern and the associated formula.

Other characterizations by Thue (1)

m_2: $0 \mapsto 130402$, $1 \mapsto 132$, $2 \mapsto 1304$, $3 \mapsto 1402$, $4 \mapsto 1404$.

h_1: $0 \mapsto 01210212021012021 2$,
$1 \mapsto 01210212$,
$2 \mapsto 01210212021$,
$3 \mapsto 01210120212$,
$4 \mapsto 0121012021$.

h_2: $0 \mapsto 0210120102012$,
$1 \mapsto 021012$,
$2 \mapsto 02101201$,
$3 \mapsto 02102012$,
$4 \mapsto 0210201$.

- $\{AA, 010, 020\}$ characterizes $h_1(m^\infty(1))$
- $\{AA, 121, 212\}$ characterizes $h_2(m^\infty(1))$

Other characterizations by Thue (2)

m_2: $0 \mapsto 130402$, $1 \mapsto 132$, $2 \mapsto 1304$, $3 \mapsto 1402$, $4 \mapsto 1404$

- $\{AA, 010, 020\}$ characterizes $h_1(m_2^\infty(1))$
- $\{AA, 121, 212\}$ characterizes $h_2(m_2^\infty(1))$

$m_2^\infty(1)$ appears in both. Interesting ?

- Take a ternary word avoiding $\{AA, 010, 020\}$, delete each letter after a 0, you get a word avoiding $\{AA, 121, 212\}$.
- $\{AA, 01, 03, 10, 12, 20, 23, 24, 31, 34, 42, 43, 141, 302, 414, 2132, 3213\}$ characterizes $m_2^\infty(1)$.

Characterizations using $m_1^\infty(0)$

$$
\begin{aligned}
h_3\colon 0 &\mapsto 001011011101110 1001,\\
1 &\mapsto 00101101101001,\\
2 &\mapsto 00010.
\end{aligned}
$$

$$
\begin{aligned}
h_4\colon 0 &\mapsto 1001001101011001101001011001001101\\
&\quad\ 10010110100110110010011010010110 0110101,\\
1 &\mapsto 100100110100101,\\
2 &\mapsto 100100110110010110100 1101.
\end{aligned}
$$

- $\{AABBCABBA, 0011, 1100\}$ characterizes $h_3(m_1^\infty(0))$.
- Containing only 10101 as a 2^+-power and 11 distinct squares characterizes $h_4(m_1^\infty(0))$.

More examples by Golnaz Badkobeh

m_3: $0 \mapsto 032$, $1 \mapsto 04$, $2 \mapsto 12$, $3 \mapsto 14$, $4 \mapsto 1432$.

h_5: $0 \mapsto 1100$, $1 \mapsto 110$, $2 \mapsto 100$, $3 \mapsto 10$, $4 \mapsto 101100$.

h_6 : $0 \mapsto 100110$, $1 \mapsto 10010110$, $2 \mapsto 0110$, $3 \mapsto 01$,
$4 \mapsto 0100110$.

- Containing only 01010 and 10101 as 2^+-powers and 8 distinct squares characterizes $h_5(m_3^\infty(0))$.
- Containing only 1001001 as a 2^+-power and 14 distinct squares characterizes $h_6(m_3^\infty(0))$.

A third example with m_3

$$\begin{aligned} h_7\colon 0 &\mapsto 00111001011000110 1,\\ 1 &\mapsto 0011100101100011010001011 0,\\ 2 &\mapsto 00111001011101,\\ 3 &\mapsto 00111001011101001110001101000101 10,\\ 4 &\mapsto 0011100101110100111000110100010110001101. \end{aligned}$$

$\{AABBCC, ABCABC, 0000, 1111, 0001110010110,$
$1110001101001, 0110100111000, 1001011000111\}$
characterizes $h_7(m_3^\infty(0))$.

Remarks

- $m_1^\infty(0)$ appears in many characterizations and we understand why.
- $m_2^\infty(1)$ appears in two similar characterizations and we see the link.
- $m_3^\infty(0)$ appears in three characterizations, seemingly by chance.

We hope to:

- Discover new useful (pure) morphic words and find their properties.
- Find links between characterizations using a same pure morphic word.
- Use them to quickly find and prove new characterizations.

Dejean words VS Pure morphic words

To obtain the avoidability index of patterns:

- I used morphic images of Dejean words.
  - Proves exponential factor complexity.
  - Automated proofs.
- Julien Cassaigne used (pure) morphic words.
  - More theoretical insights.
  - Mandatory for patterns with polynomial factor complexity.

The only word avoiding $AB.AC.BA.BC.CA$ is the fixed point Ω of $0 \mapsto 01$, $1 \mapsto 21$, $2 \mapsto 03$, $3 \mapsto 23$, up to permutation of letters.

$\Longrightarrow \{AB.AC.BA.BC.CA, 02, \cdots\}$ characterizes Ω.

One last case by Golnaz Badkobeh

m_4: $0 \mapsto 0102,$
$1 \mapsto 1013,$
$2 \mapsto 40135,$
$3 \mapsto 51024,$
$4 \mapsto 1024,$
$5 \mapsto 0135.$

h_8: $0 \mapsto 10011,$
$1 \mapsto 01100,$
$2 \mapsto 01001,$
$3 \mapsto 10110,$
$4 \mapsto 0110,$
$5 \mapsto 1001.$

Containing only 0110110 and 1001001 as 2^+-powers and 12 distinct squares characterizes $h_8(m_4^\infty(0))$.

Proof technique

To prove that $S = \mathcal{P} \cup \mathcal{F}$ characterizes a morphic word $h(m^\infty(0))$, where m is $\Sigma_{k'} \to \Sigma_{k'}$, h is $\Sigma_{k'} \to \Sigma_k$, and $k' > k$:

- Prove that extendable words in S are h-images of words over $\Sigma_{k'}$.
- Find a set $S' = \mathcal{P} \cup \mathcal{F}'$ that seem to characterize $m^\infty(0)$ over $\Sigma_{k'}$.
- Prove that S' characterizes $m^\infty(0)$ over $\Sigma_{k'}$:
  - Prove that extendable words in S' are m-images of words over $\Sigma_{k'}$.
  - Prove that the pre-images by m of words in S' are also in S'.

Remarks

- Conjecture: If $\lambda(P) = k$, then there exists a finite set S of forbidden patterns and factors containing P that characterizes a morphic word over Σ_k.
- The converse (every morphic word has a characterization) does not seem to hold: consider the Fibonacci word.
- For all these characterizations, the set can be put in the form $\{AB \cdots XAB \cdots X\} \cup \mathcal{F}$. It would be nice to have characterizations with other patterns.
- Strategy "from above" to prove avoidability.

Thank you for your attention !