Constructing Premaximal Binary Cube-free Words of Any Level

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8th International Conference on Combinatorics on Words
Prague, 2011

Repetition-free languages as trees

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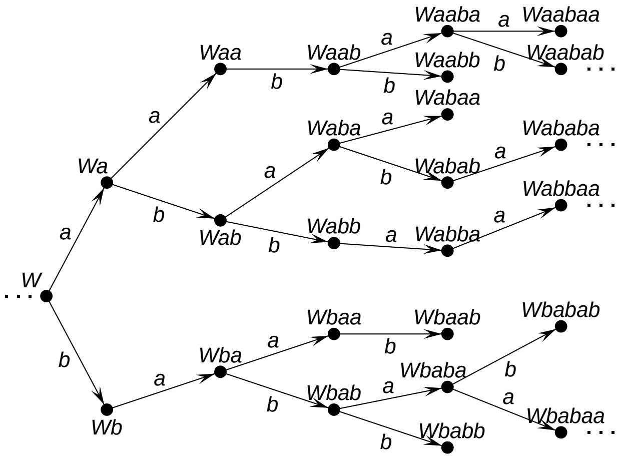

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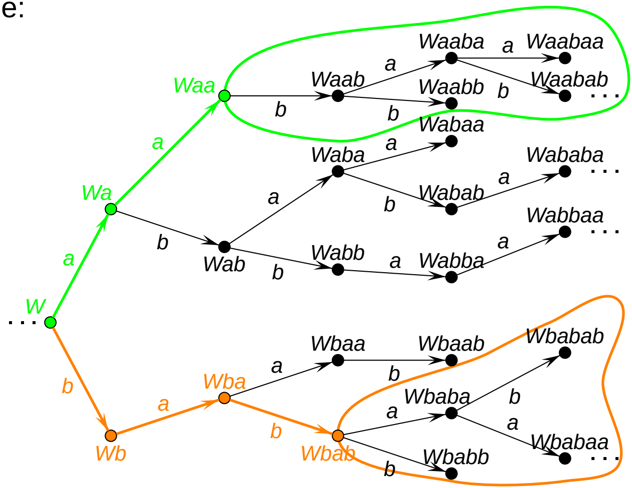

Questions and known results

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- For all kth power-free languages, the subtree generated by any word has at least one leaf [Bean, Ehrenfeucht, McNulty, 1979]
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We are going to answer question 3 for the binary cube-free language CF.

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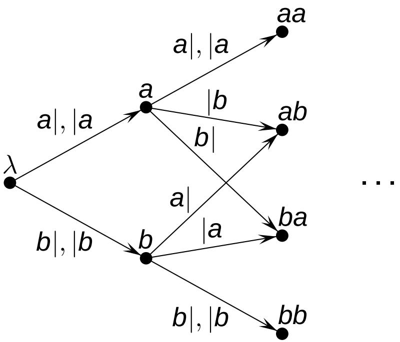

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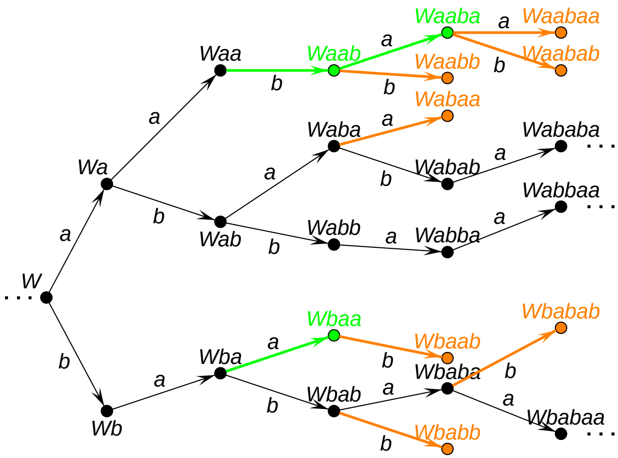
- The right counterparts of the above notions are defined in a symmetric way.
- We say that a word is maximal [premaximal] if it is both left and right maximal [respectively, premaximal]. The level of a premaximal word W is the pair $(n, k) \in \mathbb{N}$ such that n and k are the length of the longest left context of W and the length of its longest right context, respectively.

Premaximal words



Theorems

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In CF, there exist premaximal words of any level $(n, k) \in \mathbb{N}_0^2$*.*

- Moreover, the words $W_{n,k}$ of level (n, k) we have found are such that one can add a n-letter left context and a k-letter right context to $W_{n,k}$ simultaneously.

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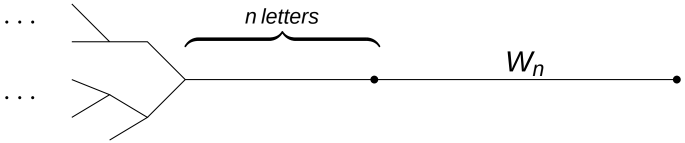
First step.

We construct an auxiliary series $\{W_n\}_0^\infty$ such that each word W_n has, up to one easily handled exception, a unique left context of any length $\geq n$.



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The basic idea for obtaining W_{n+1} with the fixed left context $\overline{x}X_n$ from the word W_n with the fixed left context X_n is to let

$$W_{n+1} = \underbrace{\quad W_n}\underbrace{xX_nW_n}\underbrace{xX_nW_n}.$$

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An attempt to build the series $\{W_n\}_0^\infty$ directly by this scheme fails because cubes will occur at the border of some words W_n and xX_n. So we insert a special "buffer" word after each of three occurrences of W_n in W_{n+1}:

$$W_{n+1} = \underbrace{\quad W_nS_n}\underbrace{xX_nW_nS_n}\underbrace{xX_nW_nS_n}$$

The series S_n

- S_n is the word inserted after W_n at the $(n+1)$th iteration;
- $S'_n = S_0 S_1 \cdots S_n$ is the factor of W_{n+1} between W_0 and the nearest occurrence of xX_n;
- S'_∞ is an image of Thue-Morse ω-word *baab abba abba baab* ... under some 108-uniform cube-free morphism
- S'_∞ consists of Thue-Morse 2-blocks *baab* and *abba*

First members of the series W_n

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$$W_4 = \quad aabaaba\,S'_3\,aabbaabaaba\,S'_3\,aabbaabaaba\,S'_3,$$

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- We do not need to build a cube-free right-infinite word similar to S'_∞, because this construction is used only once.

Děkuji mnohokrát!

Thank you!