Unambiguous 1-Uniform Morphisms

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Basic definitions

- $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\Sigma := \{a, b, c, \ldots\}$ are alphabets
- Symbols in $\mathbb{N}$ are *variables* and symbols in Σ are *letters*
- A pattern is a finite word over $\mathbb{N}$
  e. g., $2 \cdot 2 \cdot 3$
- $\mathrm{var}(\alpha)$: the set of all variables occurring in the pattern α
  e. g., $\mathrm{var}(2 \cdot 2 \cdot 3) = \{2, 3\}$

- A morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ is a mapping satisfying:

$$\sigma(\alpha \cdot \alpha') = \sigma(\alpha) \cdot \sigma(\alpha'), \text{ for all } \alpha, \alpha' \in \mathbb{N}^*$$

- For every $i \in \mathbb{N}$, $\sigma(i) \neq \varepsilon \;\Rightarrow\; \sigma$ is nonerasing
- For every $i \in \mathbb{N}$, $|\sigma(i)| = 1 \;\Rightarrow\; \sigma$ is 1-uniform

Definition

Definition

α is a fixed point of a nontrivial morphism

$\Updownarrow$

there is a morphism $\phi : \mathbb{N}^* \to \mathbb{N}^*$ satisfying $\phi(\alpha) = \alpha$ and, for a symbol x in α, $\phi(x) \neq x$

- E. g., $1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 3$ is a fixed point
- E. g., $1 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 4 \cdot 3$ is not a fixed point
- Fixed points have vital properties in various theories (decidable in polynomial time, Holub (2009))

For any alphabet Σ, for any nonerasing morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ and for any pattern $\alpha \in \mathbb{N}^+$,

- σ is unambiguous with respect to α if there is no morphism $\tau : \mathbb{N}^* \to \Sigma^*$ satisfying
  - $\tau(\alpha) = \sigma(\alpha)$
  - for some $x \in \text{var}(\alpha)$, $\tau(x) \neq \sigma(x)$
- σ is ambiguous with respect to α if σ is not unambiguous with respect to α

Example

- $\Sigma := \{a, b\}$
- $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 4 \cdot 3 \cdot 2$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a morphism satisfying

$$\sigma(x) := \begin{cases} a, & x = 1, 3 \\ b, & x = 2, 4, \end{cases}$$

$$\sigma(\alpha) = \underbrace{\overbrace{a}^{\sigma(1)}\overbrace{b}^{\sigma(2)}\overbrace{a}^{\sigma(3)}\overbrace{b}^{\sigma(4)}}_{\tau(1)}\underbrace{\overbrace{a}^{\sigma(1)}\overbrace{b}^{\sigma(4)}\overbrace{a}^{\sigma(3)}\overbrace{b}^{\sigma(2)}}_{\tau(1)} = \tau(\alpha)$$

- Thus, the morphism σ is ambiguous with respect to α

Example

- $\Sigma := \{a, b, c\}$
- $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 4 \cdot 3 \cdot 2$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a morphism satisfying

$$\sigma(x) := \begin{cases} a, & x = 1, 4 \\ b, & x = 2 \\ c, & x = 3 \end{cases}$$

$$\sigma(\alpha) = \overbrace{a}^{\sigma(1)} \overbrace{b}^{\sigma(2)} \overbrace{c}^{\sigma(3)} \overbrace{a}^{\sigma(4)} \overbrace{a}^{\sigma(1)} \overbrace{a}^{\sigma(4)} \overbrace{c}^{\sigma(3)} \overbrace{b}^{\sigma(2)}$$

- There is no other morphism τ satisfying $\tau(\alpha) = \sigma(\alpha)$. So, σ is unambiguous with respect to α

Previous literature on the existence of unambiguous morphisms

- Freydenberger, R. and Schneider (2006) show that there exists an unambiguous nonerasing morphism with binary target alphabet, with respect to a pattern α if and only if α is not a fixed point of a nontrivial morphism
- R. and Schneider (2010, 2011) investigate the existence of unambiguous erasing morphisms
- Freydenberger, N. and R. (2011) study the existence of weakly unambiguous morphisms

Main question

- Let $\alpha \in \mathbb{N}^+$ be a pattern.

  Does there exist a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α?

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Theorem (Freydenberger et al., 2006)

Let $\alpha \in \mathbb{N}^*$ be a fixed point of a nontrivial morphism, and let Σ be any alphabet. Then every nonerasing morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ is ambiguous with respect to α.

Main question

- Let $\alpha \in \mathbb{N}^+$ be a pattern that is not a fixed point of a nontrivial morphism.

  Does there exist a 1-uniform morphism $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$ that is unambiguous with respect to α?

  (For $|\text{var}(\alpha)| \leq |\Sigma|$, the answer is trivial.)

Main question

- Let $\alpha \in \mathbb{N}^+$ be a pattern that is not a fixed point of a nontrivial morphism.

  Does there exist a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$, $|\text{var}(\alpha)| > |\Sigma|$, that is unambiguous with respect to α?

Main question

- Let $\alpha \in \mathbb{N}^+$ be a pattern that is not a fixed point of a nontrivial morphism.

  Does there exist a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$, $|\mathrm{var}(\alpha)| > |\Sigma|$, that is unambiguous with respect to α?

- Fixed target alphabets: the size of Σ does not depend on the number of variables occurring in α
- Variable target alphabets: the size of Σ depends on the number of variables occurring in α

Theorem

Let $n \in \mathbb{N}$, $n \geq 4$, let Σ be an alphabet, and let

$$\alpha_n := 1 \cdot 1 \cdot 2 \cdot 2 \cdot [\ldots] \cdot n \cdot n$$

There exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α_n

$$\Updownarrow$$

$$|\Sigma| \geq 3$$

Theorem

Let

- $n \in \mathbb{N}$
- $\beta := r_1 \cdot r_2 \cdot [\ldots] \cdot r_{\lceil n/2 \rceil}$ *with* $r_i \geq 2$ *for every* i, $1 \leq i \leq \lceil n/2 \rceil$
- $\alpha := 1^{r_1} \cdot 2^{r_1} \cdot 3^{r_2} \cdot 4^{r_2} \cdot [\ldots] \cdot n^{(r_{\lceil n/2 \rceil})}$

β *is square-free*

$\Downarrow$

there exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$, $|\Sigma| = 2$, *that is unambiguous with respect to* α

Example

- $\Sigma := \{a, b\}$
- $\alpha := 1^2 \cdot 2^2 \cdot 3^3 \cdot 4^3 \cdot 5^4 \cdot 6^4$ ($\beta = 2 \cdot 3 \cdot 4$ is square-free)
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a morphism satisfying

$$\sigma(x) := \begin{cases} a, & x = 1, 3, 5 \\ b, & x = 2, 4, 6 \end{cases}$$

$$\sigma(\alpha) = \overbrace{aa}^{\sigma(1^2)} \overbrace{bb}^{\sigma(2^2)} \overbrace{aaa}^{\sigma(3^3)} \overbrace{bbb}^{\sigma(4^3)} \overbrace{aaaa}^{\sigma(5^4)} \overbrace{bbbb}^{\sigma(6^4)}$$

- σ is unambiguous with respect to α

Theorem

For every $n \in \mathbb{N}$, there exists a pattern α such that

- *α is a shortest pattern with $|\mathrm{var}(\alpha)| = n$ that is not a fixed point of a nontrivial morphism, and*
- *there exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$, $|\Sigma| = 2$ that is unambiguous with respect to α*

Example

- $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 4 \cdot 1 \cdot 5 \cdot 2 \cdot 6 \cdot 3$
- $\sigma(1) := \sigma(2) := \sigma(3) := a$ and $\sigma(4) := \sigma(5) := \sigma(6) := b$

Conjecture

Let α be a pattern with $|\text{var}(\alpha)| \geq 4$.

There exists an alphabet Σ satisfying $|\text{var}(\alpha)| > |\Sigma|$ and a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α

$$\Updownarrow$$

α is not a fixed point of a nontrivial morphism

Definition

Let

- Σ be an infinite alphabet
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a renaming

For any $i, j \in \mathbb{N}$ with $i \neq j$ and for every $x \in \mathbb{N}$, we define the morphism $\sigma_{i,j}$ by

$$\sigma_{i,j}(x) := \begin{cases} \sigma(i), & \text{if } x = j \\ \sigma(x), & \text{if } x \neq j \end{cases}$$

Conjecture

Let α be a pattern with $|\text{var}(\alpha)| \geq 4$.

There exist $i, j \in \text{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α

$$\Updownarrow$$

α is not a fixed point of a nontrivial morphism

Theorem

Let

- $\alpha \in \mathbb{N}^+$, $|\text{var}(\alpha)| > 3$
- *for every* $x \in \text{var}(\alpha)$, $|\alpha|_x = 2$
- α *not be a fixed point of a nontrivial morphism*

$$\Downarrow$$

there exist $i, j \in \text{var}(\alpha)$, $i \neq j$, *such that* $\sigma_{i,j}$ *is unambiguous with respect to* α

Example

- $\Sigma := \{a, b, c, d\}$
- $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a morphism with

  $\sigma(1) := a$, $\sigma(2) := b$, $\sigma(3) := c$ and $\sigma(4) := d$

Example

- $\Sigma := \{a, b, c, d\}$
- $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ be a morphism with

  $\sigma(1) := a$, $\sigma(2) := b$, $\sigma(3) := c$ and $\sigma(4) := d$

- The morphisms $\sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}, \sigma_{2,4}$ and $\sigma_{3,4}$ are ambiguous with respect to α
- The morphism $\sigma_{1,4}$ is unambiguous with respect to α

$$\sigma_{1,4}(x) = \begin{cases} a, & x = 1, 4 \\ b, & x = 2 \\ c, & x = 3 \end{cases}$$

$\sigma_{1,4}(\alpha) = abcaacba$

Further results

- For patterns α that contain every variable exactly *m* times and contain variables i, j such that $\alpha \neq \ldots i \cdot j \cdot \ldots \cdot j \cdot i \cdot \ldots$, we know conditions for $\sigma_{i,j}$ to be unambiguous with respect to α.
- We can construct a large set of patterns satisfying our conjecture by combining certain patterns that are not fixed points of a nontrivial morphism, under some conditions.

- Next approach to investigate the conjecture

Main idea:

- Consider words that can be morphic images of a pattern under unambiguous 1-uniform morphism, and then
- Construct suitable morphic preimages from the words

Theorem

Let

- $\alpha \in \mathbb{N}^+$ *and* Σ *be an alphabet*
- *for every* $x \in \mathrm{var}(\alpha)$, $|\alpha|_x \geq 2$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ *be a 1-uniform morphism such that, for every factor* $u_1 u_2$ *of* $\sigma(\alpha)$, $u_1, u_2 \in \Sigma$, $u_1 u_2$ *occurs in* $\sigma(\alpha)$ *exactly once*

Then

- α *is not a fixed point of a nontrivial morphism*
- σ *is unambiguous with respect to* α

- We construct patterns from De Bruijn sequences

Definition

- A non-cyclic De Bruijn sequence (of order n) is a word over a given alphabet Σ (of size k) for which all possible words of length n in Σ^* appear exactly once as factors of this sequence
- $B'(k, n)$ is the set of all non-cyclic De Bruijn sequences of order n

Example

$\Sigma := \{a, b, c\}$

- $w = aabacbbcca \in B'(3, 2)$

$$\Downarrow$$

- $\alpha = 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 3$
- $\sigma : \mathbb{N}^* \to \Sigma^*$ is an unambiguous morphism with

$$\sigma(x) = \begin{cases} a, & x = 1, 3 \\ b, & x = 2 \\ c, & x = 4 \end{cases}$$

- $\sigma(\alpha) = w$

Summary

- The existence of unambiguous 1-uniform morphisms have been investigated from two points of view, fixed target alphabets and variable target alphabets
- Our result related to fixed target alphabets imply that the characterisation of the existence of unambiguous 1-uniform morphisms might be very difficult in that case
- For variable target alphabets, we have conjectured that there exists an unambiguous 1-uniform morphism with respect to the pattern α if and only if α is not a fixed point of a nontrivial morphism
- We have shown that, for a large set of patterns, our conjecture holds true